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Optimal system and exact solutions for the generalized system of 2-dimensional Burgers equations with infinite Reynolds number

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ABSTRACT

One of the equations used for studying fluid turbulence is the 2D Burgers equations. Researchers have used various numerical methods to compute the values of the velocity components at different Reynolds number and other parameters of choice. Reynolds number is important in analyzing any type of flow when there is substantial velocity gradient. It indicates the relative significance of the viscous effect compared to the inertia effect. Results available in the literature for the 2D Burgers equations are either for laminar flow which occurs at low Reynolds numbers or for turbulent flow which occurs at high Reynolds numbers. These results cannot be used for superfluidity, the hallmark property of quantum fluids, where Reynolds number is infinite. Although Reynolds number may not be infinite in some superfluid turbulence (Barenghi, 2008; Vinen, 2005) [1,2], it is definitely the case at zero-temperature limit (Sasa and Machida, 2011) [3]. Based on this assumption, we analyse the 2D Burgers equation at infinite Reynolds number using Lie group method. Optimal system of one-dimensional subalgebras is derived and used to obtain generalized distinct exact solutions of the velocity components.

1. Introduction

Nonlinear partial differential equations (NPDEs) is a vast and active field of research which naturally modeled most real-world phenomena that occur in fields of science. Prominent among the class of NPDEs is the Burgers equation which is one of the most important equations in nonlinear evolution theory. It has been used extensively to model real world physical phenomena such as turbulence, gas dynamics, traffic flow, chemical reactions, shock-waves, nonlinear acoustic propagation, and various ecological systems. Due to its vast applicability in various fields of knowledge, researchers have used numerous methods to study its solutions analytically and numerically. Effective methods that have been used in this regard include, but not limited to, Variational iteration method [4], Adomian's decomposition method [5], Eulerian–Lagrangian method [6], modified extended Tanh-function method [7], and Lie symmetry method [8–11]. The Lie group method has had tremendous success in enabling researchers to find exact solutions of NPDEs that provide more understanding of various natural phenomena. Its powerful application to NPDEs in particular, and differential equations in general, has open numerous windows to applicability of poorly understood and complex systems.

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In this paper, we shall utilize the Lie symmetry method for the analysis of the following generalized system of 2-dimensional Burgers equation

$$\begin{aligned} &\frac{\partial u}{\partial t}+g(u)\frac{\partial u}{\partial x}+f(v)\frac{\partial u}{\partial y}=0,\\ &\frac{\partial v}{\partial t}+g(u)\frac{\partial v}{\partial x}+f(v)\frac{\partial v}{\partial y}=0, \end{aligned} \tag{1}$$

where $g(u)$ and $f(v)$ are assumed to be respective smooth functions of the velocity components $u=u(x,y,t)$ and $v=v(x,y,t)$ with non vanishing first derivatives. The system (1) is the non viscous case of the 2-dimensional Burgers equation

$$\begin{aligned} &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right),\\ &\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right), \end{aligned} \tag{2}$$

which has been studied by various researchers under appropriate boundary and initial conditions. Most of the solutions available in the literature for Eq. (2) are numerical. Values of the velocity components $u(x,y,t)$ and $v(x,y,t)$ are obtained corresponding to different choices of the Reynolds number R, the time t, and other pre-assumed parameters [4–7,12–14]. But these results are impractical in superfluid turbulence where Reynolds number is infinite. Although zero viscosity of superfluid does not automatically renders its Reynolds number infinite [1,2], it is certainly the case at zero-temperature limit [3]. Based on this fact, we exhibit various classes of generalized exact invariant solutions of the system (1). To the best of our knowledge, solutions to the system (1) has not been reported in the literature, even in the latest manuscript [15]. For the Lie symmetry method to differential equations, interested readers are refer to [16–18].

2. Group classifications

To obtain the symmetry algebra of the system (1), we take the infinitesimal generator of symmetry algebra of the form

$$X=\xi(x,y,t)\frac{\partial}{\partial x}+\tau(x,y,t)\frac{\partial}{\partial t}+\lambda(x,y,t)\frac{\partial}{\partial y}+\eta(x,y,t,u,v)\frac{\partial}{\partial u}+\varphi(x,y,t,u,v)\frac{\partial}{\partial v},$$

where $\xi(x,y,t),\ \tau(x,y,t),\ \lambda(x,y,t),\ \eta(x,y,t,u,v)$, and $\varphi(x,y,t,u,v)$ are coefficient functions to be determined. If $X^{(1)}$ is the first prolongation of the system (1), then the invariance condition

$$X^{(1)}=0$$

yields the determining equations

$$\varphi_t+g(u)\varphi_x+f(v)\varphi_y=0, \tag{3}$$

$$\eta_t+g(u)\eta_x+f(v)\eta_y=0, \tag{4}$$

$$\varphi f'(v)+f^2(v)\tau_y-\lambda_t+f(v)(-\lambda_y+\tau_t+g(u)\tau_x)-g(u)\lambda_x=0, \tag{5}$$

$$\eta g'(u)+f(v)(-\xi_y+g(u)\tau_y)-\xi_t+g(u)\tau_t-g(u)\xi_x+g^2(u)\tau_x=0. \tag{6}$$

Putting Eqs. (5) and (6) respectively into Eqs. (3) and (4) with the fact that the coefficients $\xi,\ \tau$, and λ are functions of $x,\ y,\ t$ only, we have the following:

$$\lambda_{tt}=0 \tag{7}$$

$$2\lambda_{yt}-\tau_{tt}=0 \tag{8}$$

$$\lambda_{xt}=0 \tag{9}$$

$$\lambda_{xy}-\tau_{xt}=0 \tag{10}$$

$$\lambda_{yy}-2\tau_{yt}=0 \tag{11}$$

$$\lambda_{xx}=0 \tag{12}$$

$$\tau_{xy}=0 \tag{13}$$