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On orbital-reversibility for a class of planar dynamical systems

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ABSTRACT

We give necessary conditions for the orbital-reversibility for a class of planar dynamical systems, based on properties of some invariant curves. From these necessary conditions we formulate a suitable algorithm, to detect orbital-reversibility, which is applied to a family of nilpotent systems and to a family of degenerate systems.

1. Introduction

The presence of involutory reversing symmetries (*reversibilities*) is very common in systems that occur in many disciplines of science and engineering, see e.g., [16,20]. The most frequent cases of reversibility that one finds in the literature are associated to linear involutions. In fact, they usually correspond to the called axis-reversibility (some authors call it time-reversibility), where the involution consists in the change of sign in some variable. In such cases, the presence of some reversibility is a valuable fact in the study, because each solution has a twin, obtained by changing the sign of the time variable and applying the time-reversing symmetry. This property is of utility in the understanding of the phase portrait as well as in the long time behavior of the system, being an important tool to detect periodic orbits.

In particular, for planar systems, there is a strong connection between the center problem and the reversibility property of a planar system: if the system has a non-degenerate center at the origin, then it is reversible, see [19]. The connections between centers, analytic integrability and reversibility of planar vector fields has been investigated by many authors, see [3,6,9,12,18,21], and references therein.

In this paper, we are concerned with the *orbital-reversibility* problem: a system is called *orbital-reversible* if there exists some time-reparametrization such that the resulting system admits some (nonlinear) reversibility; and our goal is to determine, in the planar case, conditions on the system to be orbital-reversible. As with the reversibility, the presence of some orbital-reversibility is useful in the understanding of the dynamical behavior of the system, because the time-reparametrizations do not change the orbits but only the speed in which they are traversed in time.

So, the orbital-reversibility property is also closely related to the center problem. For instance, the existence of an orbital reversibility in a monodromic vector field ensures the presence of a center. Also, if a planar system has a nilpotent center at the origin, then it is orbital-reversible, see [9].

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When one deals with a specific problem, the concept of orbital-reversibility is more helpful than the one of reversibility. For instance, in the study of the characterization of centers, the orbital-reversibility property is a wider concept than the reversibility; in fact, there exist nilpotent vector fields having a center that are orbital reversible but not reversible (an example is presented below).

Specifically, let us consider a planar analytic autonomous system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} = (x, y)^T \in \mathbb{R}^2, \tag{1.1}$$

having an equilibrium point located at the origin. In fact, we can assume that it is a polynomial system, that is the most frequent situation that one may find.

The problem of determining if system (1.1) has some reversibility is considered in [8], where it is shown that if system (1.1) is reversible, then it can be transformed into an axis-reversible one, by using some transformations in the state variables.

Besides transformations in the state variables, we also reparametrize the time in system (1.1). It is well known that the analytic effect of such a time-reparametrization is simple: if it is given by $\frac{dt}{dT} = \mu(\mathbf{x})$, (with $\mu(\mathbf{0}) > 0$), then transformed vector field is the original one multiplied by μ. Along this paper we will use time-reparametrizations satisfying $\mu(\mathbf{0}) = 1$, which can always be achieved by using a time-rescaling if necessary.

Next, we give a precise definition of the reversibility we will deal with:

Definition 1.1. An involution is a diffeomorphism σ such that $\sigma \circ \sigma = \mathrm{Id}, \sigma(0) = \mathbf{0}$ and $\dim(\mathrm{Fix}(\sigma)) = 1$ where $\mathrm{Fix}(\sigma) = \{\mathbf{x} \in \mathbb{R}^2 : \sigma(\mathbf{x}) = \mathbf{x}\}$ is the fixed point set of σ. It is important to remark here that the involution could be nonlinear.

We say that system (1.1) (or the vector field $\mathbf{F}$) is reversible if there exists some involution σ such that $\sigma * \mathbf{F} = -\mathbf{F}$ (we denote pull-back of a vector field of $\mathbf{F}$ by a transformation Φ as $\Phi * \mathbf{F}$). Sometimes, we make explicit the involution by saying that $\mathbf{F}$ is σ-reversible.

We say that system (1.1) (or the vector field $\mathbf{F}$) is orbital-reversible if there exist an involution σ and a scalar function μ, with $\mu(\mathbf{0}) = 1$ such that $\sigma * (\mu\mathbf{F}) = -\mu\mathbf{F}$, (this means that reparametrizing the time adequately, the system becomes reversible).

For instance, in the case of the axis-reversibility (time-reversibility), the involution is

$$R_x(x, y) = (-x, y), \text{ or } R_y(x, y) = (x, -y).$$

Notice that, in these cases, the fixed point set is a straightline (in fact, it is an axis).

As an example of the type of analysis we do here, let us consider the system

$$\dot{x} = y + xy + x^4,$$

$$\dot{y} = -x^5 + 3y^2 + 3x^3y.$$

In [8], it is shown that it is not reversible. Nevertheless, if we reparametrize the time by $\frac{dt}{dT} = \frac{1}{(1+x)^2}$, we get a system which is reversible with respect to the involution

$$\sigma(x, y) = \left(\frac{-x}{1 + 2x}, \frac{y}{(1 + 2x)^3}\right),$$

defined for $x < 1/2$. It is a simple task to show that $\mathrm{Fix}(\sigma)$ is the y-axis and, as the system is monodromic, we can conclude that the origin is a center, which is orbital-reversible but not reversible.

The underlying idea in our analysis is the reduction of the system to a pre-normal form and later to analyze the axis-reversibility modulo orbital-equivalence (see Proposition 2.10). In this way, we get necessary conditions for orbital-reversibility that provide an algorithm, discarding non-orbital-reversible cases. Hence, it is not necessary to pay attention to the convergence aspects in the normalization procedure. In other words, our analysis does not require addressing issues of convergence, so that the vector fields and scalar functions along this paper should be understood in the formal sense.

The algorithm we will use to obtain conditions for orbital reversibility is based on the following definition of orbital equivalence:

Definition 1.2. Let us consider two vector fields $\mathbf{F}, \mathbf{G}$. We say that $\mathbf{F}$ is orbital equivalent to $\mathbf{G}$ if there exist a diffeomorphism Φ and a function μ with $\mu(0) = 1$ such that $\mathbf{G} = \Phi * (\mu\mathbf{F})$.

This paper is organized as follows. In Section 2, we present necessary conditions for orbital-reversibility based on invariants of the vector field, which are used as a negative criterion for a class of planar vector fields (see Proposition 2.10). These results are applied in Section 3 to the study of the orbital reversibility for nilpotent vector fields and, in Section 4, to some vector fields having null linear part.