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The difference between a class of discrete fractional and integer order boundary value problems ☆

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ABSTRACT

In this paper, we point out the differences between a class of fractional difference equations and the integer-order ones. We show that under the same boundary conditions, the problem of the fractional order is nonresonant, while the integer-order one is resonant. Then we analyse the discrete fractional boundary value problem in detail. Then the uniqueness and multiplicity of the solutions for the discrete fractional boundary value problem are obtained by two new tools established in 2012, respectively.

1. Introduction

In this paper, we are concerned with the following discrete fractional boundary value problem

$$\begin{cases} -\Delta_{\nu-2}^{\nu} y(t) = \lambda f(t, y(t+\nu-1)), & t \in N_{0,b+1}, \\ \Delta y(\nu-2) = \Delta y(\nu+b) = 0, \end{cases} \tag{1.1}$$

where $\Delta_{\nu-2}^{\nu}$ is an discrete fractional operator, $\lambda > 0, 1 < \nu < 2, N_{0,b} := \{0, 1, 2, \ldots, b\}, b \in \mathbb{N}, b \geqslant 3$ and $f : N_{0,b+1} \times \mathbb{R} \to \mathbb{R}$.

If $\nu = 2$, then the problem (1.1) can be changed into an integer-order one:

$$\begin{cases} -\Delta^2 y(t) = \lambda f(t, y(t+1)), & t \in N_{0,b+1}, \\ \Delta y(0) = \Delta y(2+b) = 0. \end{cases} \tag{1.2}$$

In the paper, we will show that the discrete fractional boundary value problem (1.1) is nonresonant, while the integer-order one (1.2) is resonant.

Fractional calculus is a generalization of the ordinary differentiation and integration. It has played a significant role in science, engineering, economy, and other fields [1–3]. Today there are a large number of papers dealing with the continuous fractional calculus. However, the discrete fractional calculus has seen slower progress, for it is still a relatively new and emerging area of mathematics. We refer the reader to [4–13] and the references therein for the history and basic theory of the discrete fractional calculus. Of particular note is that Atici and Şengül have shown the usefulness of fractional difference equations in tumor growth modeling in [14]. We can see that it shall provide a new tool to model physical phenomena in the future. Thus, to study the fractional difference equations is meaningful, necessary and significant.

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Recent interests on the discrete fractional calculus are shown by Atici and Eloe [4,5], and in [5], they developed the commutativity properties of the fractional sum and the fractional difference operators, and first studied a class of initial value problems. Then there appeared a number of papers investigating the discrete fractional boundary value problems, such as [6,7,15–28].

In [22], the authors discussed a class of discrete fractional boundary value problems of order $\alpha \in (0,1]$

$$\begin{aligned} &-\Delta^{\alpha}_{\alpha-1}u(t) = f(t+\alpha-1, u(t+\alpha-1)), \quad t \in [0,T]_{\mathbb{Z}_0},\\ &au(\alpha-1) + bu(\alpha+T) = c. \end{aligned}$$

Their investigations have shown that differences occur between the cases of $\alpha = 1$ and $\alpha < 1$ when considering periodic boundary value conditions.

Motivated by Ferreira [22], we investigate the difference between the problems (1.1) and (1.2). Besides, We obtain the multiplicity of solutions by a new fixed point theorem by Franco et al. in [29], and prove the uniqueness by a new tool established by Jleli et al. in [30]. For more details about the discrete fractional boundary value problem and the tools used in our paper, we refer the readers to [5,11,14,16,22,29,30]. In these papers, readers cannot only find more details about the topic but also can have a more in-depth understanding.

To the best of our knowledge, most of the recent papers are concerning about the existence of solutions by the Krasnosel'shii fixed point theorem, and there are seldom papers dealing with the existence of multiple solutions. In addition, most of them dealt with the uniqueness of the solution by contraction mapping theorem. And in this paper, we will see that the uniqueness result given in our paper is a different one from those obtained by the contraction mapping theorem.

The rest of the paper is organized as follows. Section 2, we introduce some notations, definitions, and preliminary facts that will be used in the remainder of the paper. We get the Green's function $G(t,s)$ of problem (1.1) and discuss the properties of it, and then we point out the differences between problems (1.1) and (1.2) in Section 3. In Section 4 and 5, we obtain the multiplicity and uniqueness of positive solutions for the problem (1.1), respectively, and examples are given to demonstrate the applications of our results.

2. Preliminaries

Now we present some fundamental facts on the discrete fractional calculus theory which will be found in the recent papers in the literature (cf. [5–7,10,15,16]). For convenience, we introduce the following notations which will be used in the sequence:

$$\mathbb{N}_a = \{a, a+1, a+2, \ldots, \}, \quad a \in \mathbb{R},$$

$$\mathbb{N}_{c,d} = \{c, c+1, c+2, \ldots, d\}, \quad c,\ d \in \mathbb{R},\ c-d > 0,\ c-d \in \mathbb{Z}.$$

We also assume that the empty sums are zero.

Definition 2.1 [6]. Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given. Then the νth-order fractional sum of f is given by

$$(\Delta_a^{-\nu} f)(t) = \Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} f(s), \quad \text{for } t \in \mathbb{N}_{a+\nu}. \tag{2.1}$$

Also, we define the trivial sum by $\Delta_a^0 f(t) := f(t)$, for $t \in \mathbb{N}_a$.

Remark 2.1 [6]. The σ-function in Definition 2.1 comes from the general theory of time scales. It denotes the next point in the time scale after s. In this case, $\sigma(s) = s+1$, for all $s \in \mathbb{N}_a$. The term $(t-\sigma(s))^{\underline{\nu-1}}$ in Definition 2.1 is the so-called generalized falling function, defined by

$$t^{\underline{\mu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\mu)}$$

for any $t, \mu \in \mathbb{R}$ for which the right-hand side is well-defined. We appeal to the convention that if $t+1-\mu$ is a pole of the Gamma function while $t+1$ is not a pole, then $t^{\underline{\mu}} = 0$.

Definition 2.2 [6]. Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$ be given, and let $N \in \mathbb{N}$ be chosen such that $N-1 < \nu \leqslant N$. Then the νth-order fractional difference of f is given by

$$(\Delta_a^{\nu} f)(t) = \Delta_a^{\nu} f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \quad \text{for } t \in \mathbb{N}_{a+N-\nu}. \tag{2.2}$$

Remark 2.2. In [6], Holm proved that

$$\Delta_a^{\nu} f(t) = \begin{cases} \frac{1}{\Gamma(-\nu)} \sum\limits_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s), & N-1 < \nu < N, \\ \Delta^N f(t), & \nu = N. \end{cases} \tag{2.3}$$