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A note on oscillation criteria for second-order neutral dynamic equations on isolated time scales

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ABSTRACT

The principal goal of this paper is to amend oscillation results obtained in the recent paper by Saker and O'Regan (2011) [9].

1. Introduction

During the past few years, a great deal of interest in oscillatory behavior of various classes of differential equations and dynamic equations on time scales has been shown. We refer the reader to [1–4,6–9,12] and the references cited therein. Many papers deal with the oscillation of neutral equations which are often encountered in applied problems in science and technology; see, for example, Hale [5]. In particular, Saker and O'Regan [9] studied the oscillation of a class of second-order neutral dynamic equations

$$\left(p(x^{\Delta})^{\gamma}\right)^{\Delta}(t)+f(t,y(\theta(t)))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},\tag{1.1}$$

where $\mathbb{T}$ is a time scale with $\sup\mathbb{T}=\infty$ and $x:=y+r\cdot y\circ\tau$. Throughout, we assume that the following hypotheses are satisfied:

(h_1) $\gamma\geqslant 1$ is a quotient of odd positive integers, r and p are real-valued rd-continuous positive functions defined on $[t_0,\infty)_{\mathbb{T}}:=[t_0,\infty)\cap\mathbb{T}$, $\int_{t_0}^{\infty}\left(\frac{1}{p(t)}\right)^{1/\gamma}\Delta t=\infty$, and $0\leqslant r(t)<1$;
(h_2) $\tau,\theta\in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, $\tau(t)\leqslant t$, and $\lim_{t\to\infty}\tau(t)=\lim_{t\to\infty}\theta(t)=\infty$;
(h_3) $f\in \mathrm{C}([t_0,\infty)_{\mathbb{T}}\times\mathbb{R},\mathbb{R})$, $uf(t,u)>0$ for all $u\neq 0$, and there exists a positive rd-continuous function q defined on $[t_0,\infty)_{\mathbb{T}}$ such that $|f(t,u)|\geqslant q(t)|u^{\gamma}|$.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale we define the forward and backward jump operators by $\sigma(t):=\inf\{s\in\mathbb{T}|s>t\}$ and $\rho(t):=\sup\{s\in\mathbb{T}|s<t\}$, where $\inf\emptyset:=\sup\mathbb{T}$ and $\sup\emptyset:=\inf\mathbb{T},\emptyset$

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denotes the empty set. A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. Points that are right-scattered and left-scattered at the same time are called isolated. Regarding the time scales that consist of only isolated points (the so-called isolated time scales); see, for instance, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = q^{\mathbb{N}}$, and $\mathbb{T} = \mathbb{N}_0^2$, etc. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t) := f(\sigma(t))$. Some other concepts related to time scales; see Bohner and Peterson [3], Xia et al. [10], and Xia et al. [11].

Set $t_{-1} := \min_{t \in [t_0,\infty)_\mathbb{T}} \{\tau(t), \theta(t)\}$. By a solution of Eq. (1.1) we mean a function $y \in C^1_{rd}([t_{-1}, \infty)_\mathbb{T}, \mathbb{R})$ such that $x \in C^1_{rd}([t_0, \infty)_\mathbb{T}, \mathbb{R})$, $p(x^\Delta)^\gamma \in C^1_{rd}([t_0, \infty)_\mathbb{T}, \mathbb{R})$, and y satisfies (1.1) on $[t_0, \infty)_\mathbb{T}$. We consider only solutions satisfying $\sup\{|y(t)| : t \in [t_*, \infty)_\mathbb{T}\} > 0$ for all $t_* \in [t_0, \infty)_\mathbb{T}$ and tacitly assume that Eq. (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory.

To obtain oscillation criteria for (1.1), Saker and O'Regan [9] utilized a class of functions as follows: $H \in \Re$ if $H : [t_0, \infty)_\mathbb{T} \times [t_0, \infty)_\mathbb{T} \to \mathbb{R}$ and satisfies the conditions:

(i) $H(t,t) = 0,\ t \geqslant t_0,\ H(t,s) > 0,\ t > s \geqslant t_0$;
(ii) $H^{\Delta_s}(t,s) \le 0$ for $t > s \geqslant t_0$, and for each fixed $t, H^{\Delta_s}(t,s)$ is an rd-continuous function with respect to s.

In what follows, we use the notation:

$$P(t) := q(t)(1 - r(\theta(t)))^\gamma, \quad Q(t) := P(t)\alpha^\gamma(t),$$

$$P(t,T) := \int_T^t \frac{\Delta s}{p^{\frac{1}{\gamma}}(s)}, \quad \alpha(t) := \frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T) + \mu(t)},$$

$$\psi(t) := \delta^\sigma(t)[Q(t) - (p(t)a(t))^\Delta + \alpha^\gamma(t)p(t)a^{1+\frac{1}{\gamma}}(t)], \quad C(t) := (\gamma+1)\frac{\delta^\sigma(t)\alpha^\gamma(t)a^{\frac{1}{\gamma}}(t)}{\delta(t)} + \frac{\delta^\Delta(t)}{\delta(t)},$$

and

$$M(t,s) := \frac{1}{(\gamma+1)^{\gamma+1}} \frac{p(s)\delta^{\gamma+1}(s)}{(\delta^\sigma(s)\alpha^\gamma(s))^\gamma}\left[C(s) + \frac{H^{\Delta_s}(t,s)}{H(t,\sigma(s))}\right]^{\gamma+1},$$

where $a, \delta \in C^1_{rd}([t_0, \infty)_\mathbb{T}, \mathbb{R})$ and $\delta(t) > 0$.

Theorem 1.1 (*See [9, Theorem 2.1]*). *Assume that conditions (h_1)–(h_3) and $\theta(t) > t$ are satisfied. Suppose also that there exist two functions $a, \delta \in C^1_{rd}([t_0, \infty)_\mathbb{T}, \mathbb{R})$ such that, $\delta(t) > 0$ and*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t H(t,\sigma(s))[\psi(s) - M(t,s)]\Delta s = \infty \tag{1.2}$$

holds for some $H \in \Re$ and for sufficiently large T. Then every solution of Eq. (1.1) is oscillatory.

Let $\mathbb{T}$ only contain isolated points, i.e.,

$$\sigma(t) > t \quad \text{and} \quad \rho(t) < t \quad \text{for} \quad t \in [t_0, \infty)_\mathbb{T}. \tag{1.3}$$

It is not difficult to observe that $H(t,t) = 0$ such that the function M is undefined on $s = \rho(t)$. Hence, we need to change the upper limit of integral in (1.2). The objective of this paper is to solve this question. In what follows, all functional inequalities are tacitly assumed to hold for all t large enough.

2. Main results

Theorem 2.1. *Assume that conditions (h_1)–(h_3), $\theta(t) > t$, and (1.3) are satisfied. If there exist two functions $a,\ \delta \in C^1_{rd}([t_0, \infty)_\mathbb{T}, \mathbb{R})$ such that, $a(t) \geqslant 0$, $\delta(t) > 0$, and*

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^{\rho(t)} H(t,\sigma(s))[\psi(s) - M(t,s)]\Delta s = \infty \tag{2.1}$$

holds for some $H \in \Re$ and for sufficiently large T, then every solution of Eq. (1.1) is oscillatory.

Proof. Let y be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists a $t_1 \in [t_0, \infty)_\mathbb{T}$ such that $y(t) > 0,\ y(\tau(t)) > 0$, $y(\theta(t)) > 0$ for all $t \in [t_1, \infty)_\mathbb{T}$. It follows from [9, Lemma 2.1] that there exists a $T \in [t_1, \infty)_\mathbb{T}$ such that