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An open problem on the optimality of an asymptotic solution to Duffing's nonlinear oscillation problem

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ABSTRACT

Based on the renormalization group method, Kirkinis (2012) [8] obtained an asymptotic solution to Duffing's nonlinear oscillation problem. Kirkinis then asked if the asymptotic solution is optimal. In this paper, an affirmative answer to the open problem is given by means of the homotopy analysis method.

1. Introduction

Many real world problems can be reduced to the problem of solving nonlinear differential equations. Effective analytic methods such as renormalization group method and homotopy analysis method have been developed to solve these nonlinear problems.

Renormalization group (or RG) method [5,6] is an effective approach to asymptotically solve a variety of initial and boundary value problems for singularly perturbed differential equations and many related problems [4,15]. It proceeds by finding a regular perturbation solution, and then eliminating its secular terms by replacing integration constants with slowly varying functions, thereby resulting in a uniformly valid expansion.

In 2012, based on the renormalization group method, Kirkinis [8] considered the standard nonlinear oscillation problem of Duffing

$$\ddot{y} + y + \epsilon y^3 = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0, \quad \epsilon \to 0^+ \tag{1}$$

and obtained an asymptotic solution up to third order in ϵ

$$y(t,\epsilon) = 2\mathcal{R}\cos(\vartheta t) + \epsilon\frac{\mathcal{R}^3}{4}\cos(3\vartheta t) - \epsilon^2\frac{\mathcal{R}^5}{32}[21\cos(3\vartheta t) - \cos(5\vartheta t)] + \mathcal{O}(\epsilon^3) \tag{2}$$

$$\text{where} \quad \mathcal{R} = \frac{1}{2} - \epsilon\frac{1}{2^6} + \epsilon^2\frac{46}{2^{12}} + \mathcal{O}(\epsilon^3), \tag{3}$$

$$\vartheta = 1 + \epsilon\frac{3}{8} - \epsilon^2\frac{21}{256} + \epsilon^3\frac{81}{2048}. \tag{4}$$

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The first-order result of (2) recovers the multiple-scale result of Bender and Orszag [3]. At the end of the paper [8], Kirkinis asked if the asymptotic expansion (2) to the initial value problem (1) is optimal. In this paper, we give an affirmative answer to the open problem by means of the homotopy analysis method.

The homotopy analysis method (or HAM) [11–13] is a general analytic approach for seeking series solutions to nonlinear differential equations. It has been applied to solve many problems [1,9,10,17,18] in science and engineering. One of the outstanding features of the HAM is that it provides a convergence-control parameter c_0 which can be used to adjust and control the convergence regions and rates of the resulting series solutions. So the parameter c_0 can be used to deal with a lot of optimization problems [2,7,14,16].

In the popular paper [13], Liao discussed the homotopy analysis method in extensive detail via the nonlinear oscillation problem governed by

$$\ddot{y} + \lambda y + \epsilon y^3 = 0, \quad y(0) = 1, \quad \dot{y}(0) = 0, \quad \epsilon \geqslant 0, \tag{5}$$

where $\lambda \in (-\infty, +\infty)$. It is seen that the nonlinear oscillation problem (1) is only a special case of the nonlinear oscillation problem (5).

In his book in 2003 [11], Liao investigated the frequency ω of the oscillation problem (1) (with initial condition $y(0) = a$ instead of $y(0) = 1$). He obtained a second-order HAM approximation of the frequency ω with a maximum error of only 0.07% for all $\epsilon \in [0, +\infty)$.

The treatment in this paper is problem-oriented and specific. In Section 2, to solve the open problem, the HAM is applied to seek a series solution to the initial value problem (1); then by choosing an appropriate value of the parameter c_0, Kirkinis's asymptotic solution is recovered; finally by examining the averaged residual error of the approximation, the optimality of Kirkinis's asymptotic solution is verified. In Section 3, some concluding remarks are given.

2. A detailed solution

Let ω_0 be the initial guess of the frequency ω of the oscillation. Under the transformation $u(\tau) = y(t),\ \tau = \omega t$, the initial value problem (1) becomes

$$\omega^2 u''(\tau) + u(\tau) + \epsilon u^3(\tau) = 0, \quad u(0) = 1, \quad u'(0) = 0, \quad \epsilon \to 0^+. \tag{6}$$

2.1. A HAM approximation

To obtain a HAM approximation to the transformed initial value problem (6), one first constructs the zeroth-order deformation equation

$$(1-q)\mathcal{L}[\Phi(\tau;q) - u_0(\tau)] = q c_0 \mathcal{N}[\Phi(\tau;q), \Omega(q)], \tag{7}$$

with the initial guess $u_0(\tau) = \cos(\tau)$, the auxiliary linear operator

$$\mathcal{L}[\Phi(\tau;q)] = \omega_0^2 \left[\frac{\partial^2 \Phi(\tau;q)}{\partial \tau^2} + \Phi(\tau;q)\right]$$

and the nonlinear operator

$$\mathcal{N}[\Phi(\tau;q), \Omega(q)] = \Omega^2(q)\frac{\partial^2 \Phi(\tau;q)}{\partial \tau^2} + \Phi(\tau;q) + \epsilon \Phi^3(\tau;q).$$

It is seen that, as q increases from 0 to 1, $\Phi(\tau;q)$ deforms from the initial guess $u_0(\tau) = \cos(\tau)$ to the exact solution $u(\tau)$ while $\Omega(q)$ varies from the initial guess ω_0 to the exact frequency ω.

Expand $\Phi(\tau;q)$ and $\Omega(q)$ in Taylor series of q as follows:

$$\Phi(\tau;q) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau) q^m,$$

$$\Omega(q) = \omega_0 + \sum_{m=1}^{+\infty} \omega_m q^m.$$

Note that $\Phi(\tau;q)$ and $\Omega(q)$ depend on the convergence-control parameter c_0. Assuming c_0 is properly chosen so that the above series converge at $q = 1$, we have the solution series

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{8}$$

$$\omega = \omega_0 + \sum_{m=1}^{+\infty} \omega_m. \tag{9}$$