Commun Nonlinear Sci Numer Simulat 28 (2015) 138–150

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Behaviour of the extended Toda lattice

Jonathan A.D. Wattis [a,*], Pilar R. Gordoa [b], Andrew Pickering [b]

[a] School of Mathematical Sciences, University of Nottingham, University Park, Nottingham, NG7 2RD, UK
[b] Dpto. de Matemática Aplicada, ESCET, Universidad Rey Juan Carlos, C/Tulipán s/n, Móstoles 28933, Madrid, Spain

ARTICLE INFO

Article history:
Received 13 October 2014
Revised 27 March 2015
Accepted 7 April 2015
Available online 14 April 2015

Keywords:
Nonlinear dynamics
Toda lattice
Asymptotic behaviour
Self-similar behaviour

ABSTRACT

We consider the first member of an extended Toda lattice hierarchy. This system of equations is differential with respect to one independent variable and differential-delay with respect to a second independent variable. We use asymptotic analysis to consider the long wavelength limits of the system. By considering various magnitudes for the parameters involved, we derive reduced equations related to the Korteweg-de Vries and potential Boussinesq equations.

1. Introduction

In [1] an integrable non-isospectral $(2+1)$-dimensional extension of the Toda lattice hierarchy was constructed, this consisting of a sequence of pairs of equations in $p(n, t, y)$ and $q(n, t, y)$ with n being discrete and t and y continuous. The reductions of this hierarchy were found to include a $(1+1)$-dimensional differential-delay Toda lattice hierarchy, a sequence of evolution equations in $p(x, t)$ and $q(x, t)$ with both x and t continuous but where the equations involved derivatives with respect to x as well as shifts in x. It is the first member of this extended Toda lattice hierarchy that is the subject of the present paper.

In earlier papers [2,3] a $(1+1)$-dimensional differential-delay Volterra lattice hierarchy had been derived. The autonomous versions of such equations were placed within a suitable modification of the usual algebraic structure associated with completely integrable evolution equations in [4]. The first member of the $(1+1)$-dimensional differential-delay Volterra lattice hierarchy was studied in [5], where we considered various amplitudes for parameters, and obtained a number of asymptotic reductions to generalisations of the Korteweg–de Vries (KdV) equation, amongst others. In the present paper, for the first member of the extended Toda lattice hierarchy, again by considering various magnitudes for the parameters involved, we derive reduced equations related to the KdV and potential Boussinesq equations.

Section 2 contains an introduction to the relevant reduction techniques we use and the equations under study. We start by using small amplitude weakly nonlinear asymptotic techniques to reduce the Toda lattice [6] to the Boussinesq equation, and outline its reduction to the KdV equation. We also reformulate the extended Toda system [1] to make it more amenable to the asymptotic techniques used subsequently.

In Sections 3 and 4, we focus on a pair of parameters in the extended system and sequentially consider their effect on small amplitude slowly-varying solutions of the extended system. We show that when these parameters are small, the system behaves as the pure Toda lattice, whilst at larger values of these parameters other phenomena are exhibited. In Section 3 we derive various

* Corresponding author. Tel.: +44 115 951 3857.
E-mail addresses: Jonathan.Wattis@nottingham.ac.uk (J.A.D. Wattis), pilar.gordoa@urjc.es (P.R. Gordoa), andrew.pickering@urjc.es (A. Pickering).

http://dx.doi.org/10.1016/j.cnsns.2015.04.006
1007-5704/

generalisations of the potential Boussinesq equation, and in Section 4 we consider behaviour on the longer timescale, where the appropriate description is the KdV equation. Finally, in Section 5 we summarise the main results and draw conclusions.

2. Background theory – the Toda lattice and its differential-delay extension

In this section we introduce the basic Toda lattice, and recap how, through asymptotic expansions, it can be reduced to the Boussinesq equation, and the Korteweg–de-Vries equation. Finally, we introduce and reformulate the extended Toda system (a system which is both non-isospectral and differential-delay), which is the focus of the remainder of the paper.

2.1. The pure Toda system

The Toda Lattice is usually obtained from the Hamiltonian system for the Fermi–Pasta–Ulam lattice [7], with a particular choice for the interaction potential, V, namely

$$H = \sum_n \frac{1}{2} g_n^2 + V(f_{n+1} - f_n), \qquad V(\phi) = \gamma_0^2(\phi - 1 + \exp(-\phi)), \tag{2.1}$$

where $f_n(t)$ is the positions of particle n at time t and $g_n(t)$ is its momentum. The particles interact through the potential energy function $V(\cdot)$ which, in the original system studied by Fermi, Pasta and Ulam had a simple polynomial form $V(\phi) = \frac{1}{2}\phi^2 + \frac{1}{3}\alpha\phi^3$ or $V(\phi) = \frac{1}{2}\phi^2 + \frac{1}{4}\beta\phi^4$. In the Toda lattice, this potential is given by $V'(\phi) = \gamma_0^2(1 - \exp(-\phi))$. Hamilton's equations lead to

$$\frac{d^2 f_n}{dt^2} = V'(f_{n+1} - f_n) - V'(f_n - f_{n-1}) = \gamma_0^2 \exp(f_{n-1} - f_n) - \gamma_0^2 \exp(f_n - f_{n+1}). \tag{2.2}$$

The substitution $\phi(x,t) = \phi_n(t) = f_{n+1} - f_n$ with $x = n$ leads to

$$\frac{d^2 \phi_n}{dt^2} = V'(\phi_{n+1}) - 2V'(\phi_n) + V'(\phi_{n-1}). \tag{2.3}$$

The substitution $u = -\phi$ leads to

$$\gamma_0^{-2} u_{tt}(x,t) = \exp u(x+1,t) - 2\exp u(x,t) + \exp u(x-1,t) = \delta_x^2 e^{u(x,t)}, \tag{2.4}$$

where δ_x^2 is the second central difference in x. The parameter γ_0 can be eliminated by rescaling time.

The Toda soliton is given by

$$f_n(t) = F_0 + \log\left(\frac{1 - e^{-2\mu} + \eta\exp(-2\mu n + 2t\sinh\mu)}{1 - e^{-2\mu} + \eta\exp(-2\mu n - 2\mu + 2t\sinh\mu)}\right). \tag{2.5}$$

which implies

$$\exp\phi_n(t) = \exp(f_{n+1}(t) - f_n(t)) = 1 + \sinh^2(\mu)\,\mathrm{sech}^2(t\sinh\mu - \mu n + \nu), \tag{2.6}$$

for some constant wavenumber μ, and phase shift ν in e^{ϕ}, related to the phase shift η in f_n. By symmetry, $\phi_n(-t)$ is also a solution. This sech^2 shape occurs in the KdV equation as well as the Boussinesq equation. In the limit of small amplitude, that is $\mu \ll 1$, the wave is wide and travels close to the limiting speed of $c_0 = 1$.

The pure Toda system [6] can also be derived from the system

$$\begin{aligned} \widehat{p}_t(x,t) &= \gamma_0\left[\exp u\left(x + \frac{1}{2}, t\right) - \exp u\left(x - \frac{1}{2}, t\right)\right], \\ u_t(x,t) &= \gamma_0\left[\widehat{p}\left(x + \frac{1}{2}, t\right) - \widehat{p}\left(x - \frac{1}{2}, t\right)\right], \end{aligned} \tag{2.7}$$

by differentiating the latter with respect to t to eliminate $\widehat{p}$, yielding (2.4).

2.2. Small amplitude asymptotic expansion of the pure Toda system

Eq. (2.7) can be approximated using the asymptotic expansion

$$y = \epsilon x, \quad \tau = \epsilon t, \quad u(x,t) = \overline{u} + \epsilon^2 U(y,\tau), \quad \widehat{p}(x,t) = \overline{p} + \epsilon^2 P(y,\tau), \tag{2.8}$$

in which we perform a weakly nonlinear expansion of both u and p about constant solutions $u(x,t) = \overline{u}$ and $p(x,t) = \overline{p}$ to obtain

$$\epsilon^3 P_\tau = \gamma_0 e^{\overline{u}}\left[\epsilon^3 U_y + \frac{1}{24}\epsilon^5 U_{yyy} + \epsilon^5 U U_y\right], \tag{2.9}$$

$$\epsilon^3 U_\tau = \gamma_0\left[\epsilon^3 P_y + \frac{1}{24}\epsilon^5 P_{yyy}\right]. \tag{2.10}$$