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Dissipative effects in magnetohydrodynamical models with intrinsic magnetization

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ABSTRACT

A unifying non-canonical Poisson bracket has been shown to describe magnetohydrodynamic models of classical and quantum–mechanical fluids with intrinsic magnetization (or spin), and the Jacobi identity for this bracket is also proven. Their corresponding Hamiltonians are presented, and some interesting features, such as the potential absence of angular momentum conservation, are pointed out. To maintain consistency with the first and second laws of thermodynamics, a metriplectic approach to these models is highlighted which involves entropy production via a hitherto unstudied term involving the magnetization. A few promising avenues and outstanding issues for future work in this area are also discussed.

1. Introduction

It is a universally acknowledged fact that any valid physical fluid model must require an equally concrete means of dissipation. By dissipation, we do not necessarily refer to the non-conservation of energy because it can be offset in a different manner - entropy production. The first law of thermodynamics dictates the conservation of energy, whilst the second law guarantees the production of entropy. Ideally, a real-world dynamical system must possess governing equations that satisfy both these criteria. However, the most common approach involves phenomenological reasoning which may introduce effects such as 'spurious' dissipation or insufficiently motivated dynamical equations. To address these matters, it could be argued, one ought to revert to the fundamentals, viz. the action principle formulation. However, it is important to recognize that dissipative effects cannot be easily incorporated via the action principle formulation and it is easier to use a concomitant approach based on dissipative extensions of Hamiltonian systems. The Hamiltonian formulation also has several other advantages of its own, including its ability to extract knowledge of invariants (the Casimirs) and use them in equilibria and stability analyses. Cogent reviews of this methodology can be found in [1–3].

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The Hamiltonian formulation for fluids and plasmas differs from the widely known Hamilton's equations of motion in one crucial aspect – it is non-canonical in nature when one adopts Eulerian variables, which are functions of position and time. In other words, a Hamiltonian and a Poisson bracket still exist, but they are markedly different from their canonical counterparts. The difference stems from the non-canonical cosymplectic form, which changes the form of the Poisson bracket drastically, whilst preserving the properties of bilinearity, antisymmetry, the Leibniz rule and the Jacobi identity [3]. It is also degenerate, thereby giving rise to special invariants known as the Casimirs which satisfy $\{G, C\} = 0 \; \forall \; G$. The Casimirs are denoted by C, where C and G are functions (or functionals) and the symbol $\{\cdot, \cdot\}$ represents the non-canonical Poisson bracket. These Casimirs play a crucial role in determining the equilibria and their stability [4].

The non-canonical Hamiltonian approach was further extended in the 1980s through several pioneering works [5–8], which showed that dissipation could be formulated in a similar manner (to that of conservative systems) whilst satisfying the first and second laws of thermodynamics. The word 'metriplectic' was coined to describe such systems in [8]. It must be stated, however, that there are several approaches to dissipation, of which the double-bracket approach [9–11] is one of the best known, as it serves as a 'natural' extension of the non-canonical Poisson brackets. For a summary of these approaches, we refer the reader to [12].

Collectively, we have seen that there are compelling reasons for the use of Hamiltonian methods in dynamical systems, owing to their mathematical rigor and physical compatibility. Thus, we shall apply some of these approaches to two different fields of physics, both of which are MHD theories - classical ferrofluids and spin-1/2 quantum plasmas. These models are endowed with an intrinsic magnetization, which lends them an additional degree of freedom inaccessible to the usual MHD models. The field of classical ferrofluids originated in the 1960s [13], and is now widely used in condensed matter systems [14,15]. Quantum plasmas have been extensively studied in the 20th century, and we focus primarily on the spin-1/2 models developed in [16,17]. Additional details pertaining to these models can be found in several comprehensive reviews, see e.g. [18,19]. Nevertheless, we observe that the field has attracted a fair share of criticism on conceptual and practical grounds [20–22].

The outline of our paper is as follows. In Section 2, we present the non-canonical bracket and Hamiltonian for our two classes of models, and highlight their similarities and a few intriguing properties. An abbreviated proof of the Jacobi identity for this bracket is found in Appendix A. In Section 3, we present the metriplectic dynamics of classical ferrofluids, and show that entropy production also occurs via a novel magnetization term. Lastly, in Section 4, we summarize our work and indicate some promising lines of enquiry and research for future work(s).

2. Magnetohydrodynamics of fluids with intrinsic magnetization

We present an important class of magnetohydrodynamic models endowed with intrinsic magnetization, often used in modeling ferromagnetic fluids. The dynamical variables and the equations of motion are presented in Section 2.1. The Hamiltonian and the non-canonical Poisson bracket of this model are presented in Section 2.2. In Section 2.3, we indicate how quantum–mechanical corrections can be incorporated into our model. Henceforth, the paper deals entirely with 3D non-relativistic models, and we assume Cartesian geometry for the sake of simplicity; the generalization to more complex geometries is quite straightforward.

2.1. Dynamical variables and equations of motion of the model

We commence with a description of the magnetohydrodynamic model akin to the one outlined in [23]. We introduce our dynamical variables for the model:

- The mass density of the fluid ρ. The number density of each species is $n = \rho/m$, and $m = m_i + m_e$ with m_i and m_e representing the mass of the ions and electrons respectively.
- The momentum density of each species $\mathbf{M}$ defined via $\mathbf{M} = \rho\mathbf{v}$, where $\mathbf{v}$ is the center-of-mass (bulk) velocity.
- The entropy density σ, which is related to the specific entropy (per unit mass) s of the fluid via $\sigma = \rho s$.
- The magnetic field $\mathbf{B}$.
- The magnetization $\boldsymbol{\mathcal{M}}$ is the new variable. We introduce a macroscopic (ensemble-averaged) spin field $\boldsymbol{\mathcal{S}}$ related to the magnetization via $\boldsymbol{\mathcal{M}} = \mu n \boldsymbol{\mathcal{S}}$; μ denotes the magnetic moment of the particles. For electrons, note that $\mu = -\mu_B$, where μ_B is the Bohr magneton.

The magnetization of the fluid possesses a natural interpretation if one evokes Lagrangian formulation of a fluid [3]. We visualize the fluid 'particles' as endowed with an intrinsic magnetization, thereby behaving as microscopic magnets. Models with such a property are referred to as ferrofluids, and we may characterize our model as ferromagnetic magnetohydrodynamics, as the governing equation for the magnetic field is identical to the induction equation of ideal magnetohydrodynamics.

The equations for the model are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = 0, \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{v} \times \mathbf{B}\right), \tag{3}$$