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# Fast–slow dynamics in first-order initial value problems with slowly varying parameters and application to a harvested Logistic model

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ABSTRACT

In this paper, by using fast–slow decomposition and matching in singular perturbation theory, we separate the fast–slow dynamics in first-order initial value problems with slowly varying parameters and construct the asymptotic approximations to the solutions. Also we prove that the asymptotic solutions are uniformly valid on $O(1/\epsilon)$ large time interval with $O(\epsilon)$ accuracy by using the method of upper and lower solutions. As an application of the general theory, we consider a Logistic model with slowly varying parameters and linear density dependent harvest, in which, we illustrate the theoretical results through several numerical examples.



## 1. Introduction

In recent years, the harvested and unharvested Logistic models with one or more slowly varying parameters were discussed extensively. Grozdanovski et al. [1] considered the following unharvested Logistic model with slowly varying intrinsic rate of growth and carry capacity,

$$\frac{dx}{dt} = r(\epsilon t)x\left(1 - \frac{x}{k(\epsilon t)}\right), \quad x(0) = x_0. \tag{1}$$

By using a generalized two-timing method, Grozdanovski et al. [1] gave the approximate closed-form solutions to this Logistic model, which are explicit, are valid for a range of parameter values and compare well with numerically generated ones. Idlango et al. [2] and Grozdanovski et al. [3] then extended their method in [1] to the more general Logistic models, namely, the Logistic models with slowly varying parameters and density independent harvest as well as linear density dependent harvest,

$$\frac{dx}{dt} = r(\epsilon t)x\left(1 - \frac{x}{k(\epsilon t)}\right) - h(\epsilon t), \quad x(0) = x_0 \tag{2}$$

and

$$\frac{dx}{dt} = r(\epsilon t)x\left(1 - \frac{x}{k(\epsilon t)}\right) - h(\epsilon t)x, \quad x(0) = x_0. \tag{3}$$

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Grozdanovski et al. [1,3] and Idlango et al. [2] focused mainly on the constructions of the approximate closed-form solutions. They did not pay attention to the uniform validity and the error estimate on the approximate solutions.

By using slow–fast decomposition and matching in singular perturbation theory, Shen and Zhou [4] not only constructed the approximate closed-form solutions to the Logistic models (1) and (2), but also proved that the approximate solutions are uniformly valid on any bounded interval in the $t$ time scale provided that $\epsilon$ is sufficiently small. Also, they gave the error estimate between the approximate solutions and the exact solutions.

In this paper, as the generalization of the Logistic models (1)–(3), we will consider the following first-order initial value problems with slowly varying parameters, namely,

$$\frac{dx}{dt} = F(x, \epsilon t), \quad x(0) = x_0, \tag{4}$$

where $\epsilon > 0$ is a small parameter, $t$ and $x$ are respectively the independent and dependent variables and $x_0$ is the initial value.

The first-order differential equation in initial value problem (4) can be written in the following form,

$$\begin{cases} dx/dt = F(x, \lambda), \\ d\lambda/dt = \epsilon, \end{cases} \tag{5}$$

in which, we have set the variable $\lambda$ starting from

$$\lambda_0 = \lambda(0) = 0. \tag{6}$$

If this is not the case, only a linear translation is needed.

Obviously, system (5) is a slow–fast system, in which, the variable $\lambda$ can be viewed as a parameter varying slowly. This is the reason why we call (4) first-order initial value problems with slowly varying parameters. We aim to separate the fast–slow dynamics in the flows of (4) and to provide the uniformly valid asymptotic approximations to the flows when $t \in [0, O(1/\epsilon)]$, where O is a Landau order symbol and $\epsilon > 0$ is sufficiently small. Here we remark that, according to Kevorkian and Cole [5] and Verhulst [6], when $t \in [0, O(1/\epsilon)]$ with $\epsilon > 0$ sufficiently small, the constructions of the uniformly valid asymptotic approximations to the solutions of (4) is a singularly perturbed problem on infinite interval of time.

Comparing with our previous paper [4], the contributions of the present paper are twofold. Firstly, we extend the method in [4] to construct the zeroth-order asymptotic solution to the general first-order initial value problems with slowly varying parameters (4). If one wants to get the higher-order asymptotic approximation with higher accuracy, the boundary function method proposed by Vasil'eva et al. [7] is a more systematic method. Secondly, we enlarge the time interval of the uniform validity of the approximate solutions from $O(1)$ to $O(1/\epsilon)$.

The paper is arranged as follows. In the next section, by fast–slow decomposition and matching, we first construct the approximate closed-form solution to (4), and then, we prove that the approximate solution is uniformly valid when $t \in [0, O(1/\epsilon)]$. Also, we give the error estimate between the approximate solution and the exact solution by using the method of upper and lower solutions. In Section 3, as an application, we consider the Logistic model with slowly varying parameters and density independent harvest, namely, the problem defined in (3). We analyze the fast–slow dynamics of the solutions in this model and give the uniformly valid approximations to the solutions. Numerical simulations are performed to verify the theoretical results.

## 2. The main results

Under (6), the initial value problem (4) is equivalent to

$$\begin{cases} dx/dt = F(x, \lambda), \\ d\lambda/dt = \epsilon, \\ x(0) = x_0, \quad \lambda(0) = 0. \end{cases} \tag{7}$$

By defining a slow time scale $\tau = \epsilon t$ and writing (7) in terms of $\tau$, one gets

$$\begin{cases} \epsilon dx/d\tau = F(x, \lambda), \\ d\lambda/d\tau = 1, \\ x(0) = x_0, \quad \lambda(0) = 0. \end{cases} \tag{8}$$

Systems (7) and (8) are called the fast and slow systems, respectively. They are topologically equivalent each other provided $\epsilon \neq 0$.

Let $\epsilon = 0$ in (7), one gets

$$\begin{cases} dx/dt = F(x, \lambda), \\ d\lambda/dt = 0, \\ x(0) = x_0, \quad \lambda(0) = 0, \end{cases} \tag{9}$$