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Twelve limit cycles around a singular point in a planar cubic-degree polynomial system

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ABSTRACT

In this paper, we prove the existence of 12 small-amplitude limit cycles around a singular point in a planar cubic-degree polynomial system. Based on two previously developed cubic systems in the literature, which have been proved to exhibit 11 small-amplitude limit cycles, we applied a different method to show 11 limit cycles. Moreover, we show that one of the systems can actually have 12 small-amplitude limit cycles around a singular point. This is the best result so far obtained in cubic planar vector fields around a singular point.

1. Introduction

Studying bifurcation of limit cycles in planar polynomial systems is related to the second part of the well-known Hilbert's 16th problem [1]. The progress in the solution of the problem is very slow. It has not even solved the simplest quadratic systems after more than one century since the problem was posed by Hilbert at the Paris conference of the International Congress of Mathematicians in 1900. More precisely, the second part of Hilbert's 16th problem is to find the upper bound, called Hilbert number $H(n)$, on the number of limit cycles that the following system,

$$\dot{x} = P_n(x,y), \quad \dot{y} = Q_n(x,y), \tag{1}$$

can have, where $P_n(x,y)$, and $Q_n(x,y)$, represent nth-degree polynomials of x, and y. In early 1990's, Ilyashenko and Yakovenko [2], and Écalle [3] independently proved that $H(n)$ is finite for given planar polynomial vector fields. For general quadratic polynomial systems, the best result is $H(2) \geqslant 4$, obtained more than 30 years ago [4,5]. Recently, this result was also obtained for near-integrable quadratic systems [6]. However, whether $H(2) = 4$, is still open. For cubic polynomial systems, many results have been obtained on the low bound of the Hilbert number. So far, the best result for cubic systems is $H(3) \geqslant 13$ [7,8]. Note that the 13 limit cycles obtained in [7,8] are distributed around several singular points. This number is believed to be below the maximal number which can be obtained for generic cubic systems. A comprehensive review on the study of Hilbert's 16th problem can be found in a survey article [9].

In order to help understand and attack Hilbert's 16th problem the so called weak Hilbert's 16th problem was posed by Arnold [10], which is closely related to the so-called near-Hamiltonian system [11]:

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$$\dot{x} = H_y(x,y) + \varepsilon p_n(x,y), \quad \dot{y} = -H_x(x,y) + \varepsilon q_n(x,y), \tag{2}$$

where $H(x,y)$, $p_n(x,y)$, and $q_n(x,y)$, are all polynomial functions of x, and y, and $0 < \varepsilon \ll 1$, represents a small perturbation. Investigating the bifurcation of limit cycles for such a system can be now transformed to investigating the zeros of the (first-order) Melnikov function, given as an integral

$$M(h,\delta) = \oint_{H(x,y)=h} q_n(x,y)dx - p_n(x,y)dy, \tag{3}$$

along closed orbits $H(x,y) = h$ for $h \in (h_1,h_2)$, where δ denotes the parameters (or coefficients) involved in the polynomial functions q_n and p_n.

When we focus on the maximum number of small-amplitude limit cycles, $M(n)$, bifurcating from an elementary center or an elementary focus in system (1), one of the best-known results is $M(2) = 3$, which was solved by Bautin in 1952 [12]. For $n = 3$, a number of results have been obtained. Around an elemental focus, James and Lloyd [13] considered a particular class of cubic systems to obtain 8 limit cycles in 1991, and the systems were reinvestigated couple of years later by Ning et al. [14] to find another solution of 8 limit cycles. Yu and Corless [15] constructed a cubic system and combined symbolic and numerical computations to show 9 limit cycles in 2009, which was confirmed by purely symbolic computation with all real solutions obtained in 2013 [16]. Another cubic system was also recently constructed by Lloyd and Pearson [17] to show 9 limit cycles with purely symbolic computation.

On the other hand, around a center, there are also a few results obtained in the past two decades. Żołądek studied classification of cubic centers and listed 17 cases for reversible centers and 35 cases for Darboux centers [18,19]. In 1995, Żołądek [20] first proposed a rational Darboux integral,

$$H_0 = \frac{f_1^5}{f_2^4} = \frac{(x^4 + 4x^2 + 4y)^5}{(x^5 + 5x^3 + 5xy + 5x/2 + a)^4}, \tag{4}$$

and used it to prove the existence of 11 small-amplitude limit cycles around a center. This result was extensively cited by many researchers in this area. After more than ten years, another two cubic systems are constructed to show 11 limit cycles [21,22]. Recently, the system defined by (4) was reinvestigated by Yu and Han with the method of focus value computation, who only obtained 9 limit cycles [23]. This obvious difference motivated a further investigation on this problem. Very recently, Tian and Yu [24] has proved that the 11 limit cycles obtained by Żołądek [20] are not correct, and the mistakes leading to the erroneous result have been identified.

In this paper, we will consider the two cubic systems proposed by Christopher [21], and Bondar and Sadovskii [22]. The first system discussed in [21] is determined by a Darboux first integral,

$$H_1 = \frac{(xy^2 + x + 1)^5}{x^3\left(xy^5 + \frac{5}{2}xy^3 + \frac{5}{2}y^3 + \frac{15}{8}xy + \frac{15}{4} + a\right)^2}, \tag{5}$$

where a is a parameter, from which we obtain the following dynamical system,

$$\begin{aligned} \dot{x} &= 10(32a^2 - 75)^2 x(-6 - 9x - 3x^2 + 8axy - 12y^2), \\ \dot{y} &= (32a^2 - 75)^2(24a - 16ax + 90y + 15xy - 16axy^2 + 60y^3), \; (32a^2 - 75 \neq 0). \end{aligned} \tag{6}$$

System (6) has an equilibrium point, given by

$$x_e = \frac{6(8a^2 + 25)}{32a^2 - 75}, \quad y_e = \frac{70a}{32a^2 - 75}. \tag{7}$$

Shifting the equilibrium point (x_e, y_e) to the origin and setting $a = 2$ yields the system:

$$\begin{aligned} \dot{x} &= -10(342 + 53x)(289x - 2112y + 159x^2 - 848xy + 636y^2), \\ \dot{y} &= -605788x + 988380y - 432745xy + 755568y^2 - 89888xy^2 + 168540y^3, \end{aligned} \tag{8}$$

which has been studied in [21] to show 11 small-amplitude limit cycles around the origin (i.e., around the equilibrium point (x_e, y_e) of system (6)).

The second system given in [22] is described by

$$\begin{aligned} \dot{x} &= y[1 - 2r(3r^2 + 5)x + (r^2 + 3)(3r^2 + 1)^2 x^2] \equiv \tilde{f}_1(x,y), \\ \dot{y} &= -x(1 - 8rx)[1 - 3r(r^2 + 3)x] + 2[2(3r^2 - 1) - r(r^2 + 3)(15r^2 - 7)x]xy - [r(r^2 + 11) - (r^2 + 3)(3r^4 + 22r^2 - 1)x]y^2 \\ &\quad + 2r(r^2 + 3)(r^2 - 1)y^3 \equiv \tilde{f}_2(x,y), \end{aligned} \tag{9}$$

where r is a parameter. It can be shown that the origin of system (9) is a center [22].

To find the small-amplitude limit cycles bifurcating from the origin of the systems (8) and (9), in general we may apply perturbations to the systems and then compute the Melnikov functions around the loops defined by the first integral