Commun Nonlinear Sci Numer Simulat 19 (2014) 2740–2746

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

An approach to generate deterministic Brownian motion

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ARTICLE INFO

Article history:
Received 15 October 2013
Received in revised form 22 December 2013
Accepted 12 January 2014
Available online 18 January 2014

Keywords:
Brownian motion
Deterministic Brownian motion
Unstable dissipative systems
DFA analysis

ABSTRACT

We propose an approach for generation of deterministic Brownian motion. By adding an additional degree of freedom to the Langevin equation and transforming it into a system of three linear differential equations, we determine the position of switching surfaces, which act as a multi-well potential with a short fluctuation escape time. Although the model is based on the Langevin equation, the final system does not contain a stochastic term, and therefore the obtained motion is deterministic. Nevertheless, the system behavior exhibits important characteristic properties of Brownian motion, namely, a linear growth in time of the mean square displacement, a Gaussian distribution, and a −2 power law of the frequency spectrum. Furthermore, we use the detrended fluctuation analysis to prove the Brownian character of this motion.

1. Introduction

Brownian motion has been extensively studied since the findings of the biologist Brown in 1828 [1] and first described by the mathematician Thiele [2] in his paper on the least squares method published in 1880. At that time, Brownian motion was defined as a continuous-time stochastic (or probabilistic) process characterized by normal distribution. The nature of the Brownian motion is uncertain and many questions still remain open of how it could depend on particle interactions with the environment, is this process stochastic or deterministic?

After the Thiele's paper, the study of Brownian motion has been followed independently by Bachelier [3] and Albert Einstein [4], who gave the first mathematical description of a free particle Brownian motion. Later, Smoluchowski [5] brought the solution of the problem to the attention of physicists. In 1908, Langevin [6] obtained the same result as Einstein, using a macroscopically description based on the Newton's second law. He referred his approach to as "infinitely simplest" because it was much simpler than the one proposed by Einstein. Since the pioneering work of Langevin, many papers have been devoted to the description of Brownian motion [7–16], where characteristic features of this behavior have been defined.

The dynamical model of Brownian motion provided by Langevin [6], who used a second-order differential equation with a stochastic term, seems apparently from the nature of randomness. On the other hand, it is widely believed that Brownian motion can be rigorously derived from totally deterministic Hamiltonian models of classical mechanics. One of the reasons for this conviction is related to the widely used Van Hove's method [17–19]. In one way or another, many attempts to establish a unified view of mechanics and thermodynamics [20] traced back to the Van Hove's approach. The result of their method depended on whether one adopted the Heisenberg perspective corresponding to the time evolution of observables,

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http://dx.doi.org/10.1016/j.cnsns.2014.01.010

or the Schrödinger perspective corresponding to the time evolution of the Liouville density. In [21] a Fokker–Planck equation has been derived with the aid of a set of variables of interest interacting with a booster, i.e., a dynamical system mimicking the action of an ideal thermostat with no need of ad hoc statistical assumptions; this approach is based on the assumption of a large number of degrees of freedom, the booster is an n-dimensional deterministic system. In the former case, the usual outcome was derived from the ordinary Langevin equation.

The idea of deterministic Brownian motion has been also moot in hydrodynamics and oscillatory chemical reactions, where in spite of an erratic or random character of time evolution, the observed motion is completely deterministic and sometimes it is referred to as *microscopic chaos* [22–26]. In 1998 Gaspard, et al. [27] have reported on the experimental evidence of microscopic chaos in fluids, obtained by direct observation of Brownian motion of a colloidal particle suspended in water. Deterministic random walk of a phase difference, similar to Brownian motion, has also been observed in coupled chaotic oscillators [28]. A deterministic Brownian motion generator has been previously proposed by Trefàn et al. [29], where the nonlinear generator has been presented by a discrete system which generates pseudo-random numbers [30]. The microscopic chaotic process drives a Brownian particle and has "statistical" properties that differ markedly from the standard assumption of Gaussian statistics.

In many paper devoted to Brownian motion, this behavior is characterized by specific properties, such as a linear in time growth of the mean square displacement, an exponential in time decay of the positional autocorrelation function, and the Lorentzian shape of the power spectrum with a -2 power law of a high-frequency slope [19,27,31]. Another important way to determine Brownian motion is the *detrended fluctuation analysis* (DFA) developed by Peng et al. [32]. The DFA allows one to measure a simple quantitative parameter, the scaling exponent β_v which characterizes correlation properties of a signal.

In this paper we introduce an approach to generate deterministic Brownian motion and determine its character by analyzing time series, power spectrum, and via DFA.

2. Model

A typical example of Brownian motion is particle mixing agitation in fluids. The perpetual motion of a particle occurs due to collisions with molecules of the surrounding fluid. Under normal conditions in a liquid, a Brownian particle suffers from about 10^{21} collisions per second, this is so frequent that we cannot really speak of separate collisions. Furthermore, since each collision can be thought of as producing a kink in the path of the particle, one cannot hope to follow the path in any detail, i.e., the details of the path are infinitely fine. Each of these collisions is always determined by the last event produced by physical interactions in the system.

The modern theory of Brownian motion of a free particle (in the absence of an external field of force) is generally governed by the Langevin equation [6]

$$\ddot{x} = -\gamma\dot{x} + A_f(t), \tag{1}$$

where $\dot{x} = dx/dt$ and $\ddot{x} = d^2x/dt^2$ denote the particle velocity and the acceleration, respectively. According to this equation, the influence of the surrounding medium on the particle motion can be split into two parts. The first term $-\gamma\dot{x}$ stands for the dynamical friction applied to the particle and the second term $A_f(t)$ is the fluctuation acceleration which provides a stochastic character of Brownian motion and depends on the fluctuation force $F_f(t)$ as $A_f(t) = F_f(t)/m$, where m is the particle mass.

It is assumed that the friction term $-\gamma\dot{x}$ is governed by the Stokes' law which states that the friction force $6\pi a\eta\dot{x}/m$ decelerates a spherical particle of radius a and mass m. Hence, the friction coefficient is

$$\gamma = 6\pi a\eta/m, \tag{2}$$

where η denotes the viscosity of the surrounding fluid.

Concerning the fluctuation term $A_f(t)$, we make two principal assumptions:

(i) $A_f(t)$ is independent of $\dot{x}$.
(ii) $A_f(t)$ varies extremely fast as compared with the variation of $\dot{x}$.

The latter assumption implies that there exists a time interval Δt during which the variations in $\dot{x}$ are very small. Alternatively, we may say that though $\dot{x}(t)$ and $\dot{x}(t+\Delta t)$ are expected to differ by a negligible amount, no correlation between $A_f(t)$ and $A_f(t+\Delta t)$ exists.

Because a particle is immersed in a liquid or gas at ordinary pressure, Einstein [4] used the Stokes formula to calculate the mean square $\overline{\Delta x^2}$ of displacement Δx of a spherical particle in a given direction x after a given time τ to be

$$\overline{\Delta x^2} = 2Dt = \frac{RT}{N}\frac{1}{3\pi\eta a}\tau, \tag{3}$$

where $\overline{\Delta x^2} = \overline{x^2} - \overline{x_0^2}$, D is the diffusion coefficient at temperature T, R is the gas constant, and N is the Avogadro number. Brownian motion occurs in systems where the mechanisms governing energy dissipation are distinct from those of energy storage [27,19,31]. In Brownian motion, the mean square displacement at short times grows linearly with time, i.e. $\overline{\Delta x^2} \propto t^{\mu}$.