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Lagrangian descriptors for two dimensional, area preserving, autonomous and nonautonomous maps

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ABSTRACT

In this paper we generalize the method of Lagrangian descriptors to two dimensional, area preserving, autonomous and nonautonomous discrete time dynamical systems. We consider four generic model problems – a hyperbolic saddle point for a linear, area-preserving autonomous map, a hyperbolic saddle point for a nonlinear, area-preserving autonomous map, a hyperbolic saddle point for linear, area-preserving nonautonomous map, and a hyperbolic saddle point for nonlinear, area-preserving nonautonomous map. The discrete time setting allows us to evaluate the expression for the Lagrangian descriptors explicitly for a certain class of norms. This enables us to provide a rigorous setting for the notion that the "singular sets" of the Lagrangian descriptors correspond to the stable and unstable manifolds of hyperbolic invariant sets, as well as to understand how this depends upon the particular norms that are used. Finally we analyze, from the computational point of view, the performance of this tool for general nonlinear maps, by computing the "chaotic saddle" for autonomous and nonautonomous versions of the Hénon map.

1. Introduction

Lagrangian descriptors (also referred to in the literature as the "*M* function") were first introduced as a tool for finding hyperbolic trajectories in [16]. In this paper the notion of *distinguished trajectory* was introduced as a generalization of the well-known idea of distinguished *hyperbolic* trajectory. The numerical computation of distinguished trajectories was discussed in some detail, and applications to known benchmark examples, as well as to geophysical fluid flows defined as data sets were also given. Later [18] showed that it could be used to reveal Lagrangian invariant structures in realistic fluid flows. In particular, a geophysical data set in the region of the Kuroshio current was analyzed and it was shown that Lagrangian descriptors could be used to reveal the Lagrangian skeleton of the flow, i.e. hyperbolic and elliptic regions, as well as the invariant manifolds that delineate these regions. A deeper study of the Lagrangian transport issue associated with the Kuroshio using Lagrangian descriptors is given in [19]. Advantages of the method over finite time Lyapunov exponents (FTLE) and finite size Lyapunov exponents (FSLE) were also discussed.

Since then Lagrangian descriptors have been further developed and their ability to reveal phase space structures in dynamical systems more generally has been confirmed. In particular, Lagrangian descriptors are used in [4] to reveal the Lagrangian structures that define transport routes across the Antarctic polar vortex. Further studies of transport issues related to the Antarctic polar vortex using Lagrangian descriptors are given in [5] where vortex Rossby wave breaking is

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related to Lagrangian structures. In [25] Lagrangian descriptors are used to study the influence of coherent structures on the saturation of a nonlinear dynamo. In [21] Lagrangian descriptors are used to analyze the influence of Lagrangian structure on the transport of buoys in the Gulf stream and in a region of the Gulf of Mexico relevant to the Deepwater Horizon oil spill. In [17] a detailed analysis of the behavior of Lagrangian descriptors is provided in terms of benchmark problems, new Lagrangian descriptors are introduced, extension of Lagrangian descriptors to 3D flows is given (using the time dependent Hills spherical vortex as a benchmark problem), and a detailed analysis and discussion of the computational performance (with a comparison with FTLE) is presented.

Lagrangian descriptors are based on the integration, for a finite time, along trajectories of an intrinsic bounded, positive geometrical and/or physical property of the trajectory itself, such as the norm of the velocity, acceleration, or curvature. Hyperbolic structures are revealed as singular features of the contours of the Lagrangian descriptors, but the sharpness of these singular features depends on the particular norm chosen. These issues were explored in [17], and further examined in this paper.

All of the work thus far on Lagrangian descriptors has been in the continuous time setting. In this article we generalize the method of Lagrangian descriptors to the discrete time setting of two dimensional area preserving maps, both autonomous and nonautonomous, and provide theoretical support for their performance.

This paper is organized as follows. In Section 2 we defined discrete Lagrangian descriptors. We then consider four examples. In Section 2.1 we consider a linear autonomous area preserving map have a hyperbolic saddle point at the origin, in 2.2 we consider a nonlinear autonomous area preserving map have a hyperbolic saddle point at the origin, in 2.3 we consider a linear nonautonomous area preserving map have a hyperbolic saddle trajectory at the origin, and in 2.4 we consider a nonlinear nonautonomous area preserving map have a hyperbolic trajectory at the origin. For each example we show that the Lagrangian descriptors reveal the stable and unstable manifolds by being singular on the manifolds. The notion of "being singular" is made precise in Theorem 1. In Section 3 we explore further the method beyond the analytical examples. We use discrete Lagrangian descriptors to computationally reveal the chaotic saddle of the Hénon map, and in Section 4 we consider a nonautonomous version of the Hénon map. In Section 5 we summarize the conclusions and suggest future directions for this work.

2. Lagrangian descriptors for maps

Let

$$\{x_n, y_n\}_{n=-N}^{n=N}, \quad N \in \mathbb{N}, \tag{1}$$

denote an orbit of length $2N+1$ generated by a two dimensional map. At this point it does not matter whether or not the map is autonomous or nonautonomous. The method of Lagrangian descriptors applies to orbits in general, regardless of the type of dynamics that generate the orbit.

The first Lagrangian descriptor (also known as the "M function") for continuous time systems was based on computing the arclength of trajectories for a finite time [16]. Extending this idea to maps is straightforward, and the corresponding discrete Lagrangian descriptor (DLD) is given by:

$$MD_2 = \sum_{i=-N}^{N-1} \sqrt{(x_{i+1}-x_i)^2 + (y_{i+1}-y_i)^2}. \tag{2}$$

In analogy with the work on continuous time Lagrangian descriptors in [17], we consider different norms for the discretized arclength as follows:

$$MD_p = \sum_{i=-N}^{N-1} \sqrt[p]{|x_{i+1}-x_i|^p + |y_{i+1}-y_i|^p}, \quad p > 1, \tag{3}$$

and

$$MD_p = \sum_{i=-N}^{N-1} |x_{i+1}-x_i|^p + |y_{i+1}-y_i|^p, \quad p \leqslant 1. \tag{4}$$

Considering the space of orbits as a sequence space, (3) and (4) are the ℓ^p norms of an orbit.

Henceforth, we will consider only the case $p \leqslant 1$ since the proofs are more simple in this case. Now we will explore these definitions in the context of some easily understood, but generic, examples.

2.1. Example 1: a hyperbolic saddle point for linear, area-preserving autonomous maps

2.1.1. Linear saddle point

Consider the following linear, area-preserving autonomous map: