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Detecting regular dynamics from time series using permutations slopes

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In this paper we present the entropy related to the largest slope of the permutation as an efficient approach for distinguishing between regular and non-regular dynamics, as well as the similarities between this method and the three-state test (3ST) algorithm. We theoretically establish that for suitably chosen delay times, permutations generated in the case of regular dynamics present the same largest slope if their order is greater than the period of the underlying orbit. This investigation helps making a clear decision (even in a noisy environment) in the detection of regular dynamics with large periods for which PE gives an arbitrary nonzero complexity measure.

1. Introduction

Complexity measure is important as it allows comparing time series and distinguishing between regular (e.g. periodic) and non-regular behaviors. A deterministic dynamical system generating non-regular dynamics is said to be chaotic. Detecting chaos from an arbitrary series of observations remains a challenging task [1–6], as it is difficult to make a clear difference between the deterministic chaotic and stochastic dynamics. Some investigations have been carried out in this area and are still giving promising results [7–9]. If the system is assumed to be deterministic, measuring its complexity is useful for determining whether its behavior is predictable or not. Entropies, fractal dimension and Lyapunov exponent (LE) are some examples of complexity parameters.

Particular interest has been reserved to entropies as some of them can be directly applied to the series of observations [1,10–14]. In this perspective, Bandt and Pompe have proposed the permutation entropy (PE) [1], which is actually widely used in many fields due to its conceptual and computational simplicity. The PE is based on the ordinal pattern analysis and is easily calculated for any type of time series, be it regular, chaotic, noisy, or reality based. It has been successfully applied to the study of structural changes in time series and the underlying system dynamics [15–19]. In addition to its robustness against noise, it has been verified that the PE behaves similar to the largest Lyapunov exponent and can therefore be used for the detection of chaos in dynamical systems [20].

However, although regular dynamics present vanishing or negligible complexities, there is no particular value or property of the PE for the characterization of regular dynamics as it is the case for the largest LE, which makes it less suitable for chaos detection. Indeed, in some examples given on chaos detection, PE tracks the largest LE with a uniform bias that depends on

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the underlying system and the parameter setting of the PE algorithm: even perfectly predictable dynamics are characterized by a nonzero entropy. The dependence on the uniform bias can be sometimes difficult to determine when dealing with an unknown single time series. Despite the modification proposed by the weighted PE [21] and the modified PE [22] algorithms to overcome some shortcomings of the PE, no solution has been proposed to address this concern. For the PE algorithm to be efficiently implemented, a principle based on the use of lookup tables was presented in [23]. Defining lookup tables for large permutation order n is difficult as the number of permutations is equal to $n!$. In the case of the modified PE, the number of permutations is given by the *ordered Bell numbers*, which is greater than $n!$ [23]. Thus approximating the Kolmogorov–Sinai (KS) entropy from the PE is quite difficult as it requires large n. Moreover, defining a lookup table may not be useful if the algorithm is implemented for embedding systems. Recently the conditional entropy of ordinal patterns was proposed that provides more reliable estimation of the KS entropy [24] than the PE.

In 2004, Gottwald and Melbourne proposed the 0–1 test for chaos detection from time series. The test presents the advantage to be binary as it outputs 0 for regular dynamics and 1 for non-regular dynamics. The 0–1 test has shown competitive results and has been successfully applied to many types of dynamical systems and experimental data [25–27]. The test is still in improvement and has been recently slightly modified for an efficient application to strange non-chaotic attractors (SNA) [28]. The 0–1 test is sensitive to the sampling frequency. Gottwald and Melbourne showed that in the case of continuous time systems, it fails to detect chaos in oversampled time series, hence it is necessary to reduce the sampling frequency to the Nyquist frequency. However, such a condition is not consistent with the digital signal processing requirement for which the sampling frequency needs to be greater than the Shannon limit. In order to overcome such a limiting property, we proposed the modified 0–1 test in which the 0–1 test is applied to the local maxima and minima of the observations, instead of directly applying to the entire observation [29]. The modified 0–1 test thus allows to easily detect chaos from oversampled time series. However, despite this improvement, the 0–1 test remains computationally costly and cannot be used for real-time analysis of time series as it is the case for the PE. Moreover, the calibration of the test sometimes depends on the system under study.

Without prior knowledge on the PE, we proposed another approach for time series analysis, namely the three-state test (3ST) for chaos detection in discrete maps, which also belongs to the group of ordinal pattern analysis methods [30]. The 3ST presents the advantage to perform both the detection of the regularity or non-regularity and the period estimation in time series. The difference between the PE and the 3ST comes from the statistical exploitation of the permutations. Indeed, instead of constructing ordinal patterns (permutations) of fixed order n like in the PE, in the 3ST, data sequences are ordered using different values of n and the corresponding permutations are studied. By this approach, no probability is computed as the permutations do not have the same length. Moreover, the permutation list may be very large, depending on the length of the time series, hence it has memory and computationally expensive. For this purpose, each permutation is replaced by its largest slope S. The 3ST can easily detect the period-doubling route and output the corresponding periods as discrete numbers (periods of stable limit-cycles) [30]. In addition, as an ordinal method for time series analysis, the 3ST is also computationally low cost and was designed for possible real-time applications. Recently, we proposed an improvement of the 3ST for clear discrimination between periodic, quasi-periodic and chaotic dynamics [31]. We thus defined λ_P as the sensitivity of the 3ST chaos indicator, namely λ, to the initial phase. We also showed that λ_P is equivalent to computing λ using permutations with fixed order [31]. By this definition, the 3ST and the PE appear closer, even if only the largest slopes of permutations and no probabilistic approach are used in the 3ST algorithm. However, the fundamental question is to know whether the use of the largest slopes is reliable for chaos detection.

In this paper, we theoretically prove the usefulness of the permutation slopes for the discrimination between regular and non-regular dynamics. We further establish the relationship between the 3ST and the PE by computing the entropy related to the permutations largest slopes, and show that it can be efficiently applied to the detection of chaos in dynamical systems.

2. Mathematical fundamentals

2.1. Usefulness of the permutations slopes

Let $\{x_t\}_{t=1,\dots,T}$ be a time series of length T where t is the time index. The PE of order n is defined as a measure of the probabilities of permutations of order n [1]. Permutations of order n are obtained from the comparison of neighboring values (increasing order) in embedding vectors $\mathbf{x}_t = (x_{t+1}, x_{t+1+\tau} \dots, x_{t+1+m\tau}, \dots, x_{t+1+(n-1)\tau})$, where n is the embedding dimension (number of values in $\mathbf{x}_t$), τ the distance between two values in $\{x_t\}$ or delay time of samples and $m+1$ the index of $x_{t+1+m\tau}$ in $\mathbf{x}_t$, $m \in \mathbb{N}$. Let P_t be the permutation derived from $\mathbf{x}_t$. $P_t = \binom{1,2,3,\dots,n}{5,n,1,\dots,3}$, for example is obtained by sorting the values of $\mathbf{x}_t$ in ascending order, with $x_{t+5} < x_{t+n} < x_{t+1} < \dots < x_{t+3}$. Identical values are sorted by the ascending order of their time index. The permutation entropy of order n is thus given by

$$H(n) = -\sum p(\theta) \cdot \ln(p(\theta)) \tag{1}$$

where

$$p(\theta) = \frac{\#\{t | t \leqslant T-n, P_t = \theta\}}{T-n+1} \tag{2}$$