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A fast numerical method for the valuation of American lookback put options

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ABSTRACT

A fast and efficient numerical method is proposed and analyzed for the valuation of American lookback options. American lookback option pricing problem is essentially a two-dimensional unbounded nonlinear parabolic problem. We reformulate it into a two-dimensional parabolic linear complementary problem (LCP) on an unbounded domain. The numeraire transformation and domain truncation technique are employed to convert the two-dimensional unbounded LCP into a one-dimensional bounded one. Furthermore, the variational inequality (VI) form corresponding to the one-dimensional bounded LCP is obtained skillfully by some discussions. The resulting bounded VI is discretized by a finite element method. Meanwhile, the stability of the semi-discrete solution and the symmetric positive definiteness of the full-discrete matrix are established for the bounded VI. The discretized VI related to options is solved by a projection and contraction method. Numerical experiments are conducted to test the performance of the proposed method.

1. Introduction

Options as a kind of financial derivatives, has been very popular in European and American countries, and which will be launched in China at the end of this year. The research interests on the option pricing problem gradually spread all over the world. For the classical European and American options, there exist extensive results on theoretical analysis and numerical method (cf. [3,8,10,18,22,30–32]). In this paper, we mainly concern on the valuation of American lookback options, which is more complicated and widely applied than classical American options. Lookback options is first proposed by Goldman et al. [17] in 1979, which is a kind of path-dependent options. The payoff function of lookback options not only depends on the on-the-spot underlying asset price, but also on the extreme value of the underlying asset price over the history, e.g., fixed strike lookback call options, floating strike lookback put options (cf. [4]). Generally speaking, like classical options, lookback options also can be divided into European and American. For European lookback options, a closed-form solution was given in [9] by using the well-known Black–Scholes equation (cf. [5,20]) associated with the initial and boundary value conditions. Due to the computational cost of the analytical solution is too high, there also exist some researches on numerically solving European lookback options (cf. [2,19]). For American lookback options, the research results are not too much. At first, investigators try to give a closed-form solution as European. For example, in 2004, Lai and Lim [26,27] obtain an analytic price formula for American lookback options using a decomposition, which expresses the price as the sum of the corresponding European value and an early exercise premium determined by the optimal exercise boundary. However the optimal exercise

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boundary (cf. [12,24,25]) was not known actually in practice. Therefore, the approach on searching for analytical solution was blocked, and the numerical method for American lookback options has attracted increasingly more interest.

In the past two decades, some numerical methods, such as the binomial method, Monte Carlo/Quasi-Monte Carlo method, finite difference and finite element method, etc., have been proposed for the valuation of American lookback options, for instance, [1,6,7,11,13–16,28]. In this paper, we only emphasize some works related to our algorithm. In 2001, a finite difference method was proposed by Yu et al. to compute the exercise boundary of American lookback options (cf. [29]). In the same year, Lai and Lim [26,27] introduced the Bernoulli walk approach to solve the American lookback option pricing problem. In the book [23], Jiang reformulated the option pricing problem to be a linear complementary problem, and mentioned that the linear complementary problem could be solved by the binomial method. This book also proved that the binomial method was a special case of the finite difference method. Based on the ideas of [23], Zhang et al. [33] proposed a new algorithm, which discretes the pricing problem by a finite difference method and solves the discretized system by SOR iteration. Finite element methods have also been used in this field for its solid theoretical framework, refer to [13,14].

This paper is mainly concerned on developing a fast and accurate numerical algorithm for American lookback option pricing problems. For simplicity, we only consider the floating strike lookback put options, the call options can be treated in a similar way. By analyzing the previous researches on American lookback options, we summarize two challenges for solving this problem: (1) American lookback put option pricing problem is a two-dimensional nonlinear parabolic problem on an unbounded domain, which could not be solved directly by numerical methods. The technique to reformulate it into a simple bounded problem is our first concern. (2) The other challenge is how to design an efficient algorithm for solving the resulting simple bounded problem. As we all known that a fast and efficient algorithm is very important in financial field, especially for the products related to stocks.

As regards the former challenge, we first introduce the linear complementary problem corresponding to American lookback put options. Then the widely used numeraire transformation (cf. [23,33]) is applied to reduce the problem to a one-dimensional problem in space. For the unbounded domain, using the property of the optimal exercise boundary, Zhang et al. [33] present a truncation technique for no dividend options. We shall follow the idea of [33] to deal with the general lookback options. Thus, the lookback option pricing problem has become a one-dimensional bounded linear complementary problem.

The techniques for dealing with the latter challenge are our main results. In this paper, we firstly reformulate the one-dimensional bounded linear complementary problem related to American lookback options into a bounded variational inequality (VI), and then discretize the bounded VI by a finite element method. Furthermore, the stability of the semi-discrete solution and the symmetric positive definiteness of the full-discrete matrix are established for the bounded VI. At last, the discretized system is solved by a projection and contraction method (cf. [21]) rapidly and accurately.

Compared to other methods, the main advantage of the proposed method in this paper is that the computing speed is much more faster under the same given accuracy, which is verified in the numerical simulations.

The rest of this paper is organized as follows. In Section 2, we shall describe the Black–Scholes model, its corresponding linear complementary problem and variational inequality form for American lookback put options. The numeraire transformation and domain truncation condition will be recalled and employed to reduce the space dimension and size. In Section 3, a finite element method is applied to discretize the one-dimensional bounded variational inequality. We prove that the semi-discrete solution is stable and the full-discrete matrix is a symmetric positive definite one. A projection and contraction algorithm for solving the resulting discretized system is also proposed in this section. In Section 4, numerical simulations are presented to test the performance of the proposed method and to compare it with the binomial method. Some conclusive remarks are given in Section 5.

2. Pricing models via transformation

In this section, the Black–Scholes model for American lookback put options is presented, which is a two-dimensional parabolic problem on an unbounded domain. We reform it into a two-dimensional unbounded parabolic linear complementary problem (LCP). By using the numeraire transformation and some given information on the optimal exercise boundary (cf. [23]), we reduce the two-dimensional unbounded parabolic LCP to a one-dimensional bound-ed one. At the end of this section, the variational inequality form relevant to the one-dimensional bounded parabolic LCP is provided, which will be used for numerical computation. For brevity, we only consider American lookback put options since American lookback call options can be treated in a similar way.

2.1. Black–Scholes model

Let us consider the classic Black–Scholes model for an American lookback put option. Assume $S,\ t,\ \sigma,\ r,\ q$ and T denote the price of a certain amount of underlying asset, time, volatility, interest rate, dividend rate and maturity date, respectively, and J stands for the peak value of S attained from time zero to time t. The value $P(S,J,t)$ of an American lookback put option satisfies the following free boundary value problem (cf. [23]):