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Symmetry analysis of reaction diffusion equation with distributed delay

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ABSTRACT

This paper study the reaction–diffusion equation with distributed delay from the Lie group theoretic point of view. We give at first the evolutionary infinitesimal vector field $\mathbf{v}$ and a number of group invariant solutions corresponding to $\mathbf{v}$ by general symmetry group theory.

1. Introduction

Reaction–diffusion equations with time delays appear in problems with delaying links where certain information processing is needed, such as in population dynamics, biology and epidemiology. During the past 30 years, many researchers have paid attention to reaction–diffusion equations with different kinds of time delays. According to the difference of models and the impact of different factors which were take into account in the models, time delays mainly divided into the following form: discrete time delay (see, e.g., [2,19,7] and references therein), distributed time delay (see, e.g., [1,3] and references therein), both time delay and spatially nonlocal effect (see, e.g., [6,10,21] and references therein) and spatiotemporal time delay (see, e.g., [4,17,20] and references therein). It is noteworthy that the references mentioned above mainly studied the existence of traveling wave solutions.

Lie group theory is one of a powerful and direct approach to construct exact solutions of nonlinear differential equations [9,16]. Around in the middle of nineteenth century, Norwegian mathematician Sophus Lie found the invariance of differential equations under a continuous group of symmetries. The applications of Lie's continuous symmetry groups, i.e., Lie group, include diverse fields, such as algebraic topology, differential geometry, numerical analysis, bifurcation theory and so on. Roughly speaking, a symmetry group of system of differential equations is a group which transforms solutions of the system to other solutions. For a system of partial differential equations, we can use general symmetry groups to explicitly determine a special type of solutions which are themselves invariant under one-parameter symmetry group of the system. The group-invariant solutions are found by solving a reduced system of differential equations involving fewer independent variables than the original one.

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We should note that Lie group theory, developed for partial differential equations, can not be directly applied to delay differential equations [13,14]. The main obstacle is their nonlocality: the nonlocality does not allow using the manifold's approach to define a Lie group.

There have been few investigations of delay differential equations by group approach up to now. It should be mentioned that Linchuk [5] suggested a group approach to research functional differential equations, where delay differential equations are replaced by an underdetermined differential equations for which the classical group analysis is applied. But the extension of equations narrows a set of admitted Lie groups. After then Tanthanuch and Meleshko [13,14] generalized the group analysis theory to delay differential equation, defining an admitted Lie group and determining equation for delay differential equation. Notice that this definition does not need that the admitted Lie group should transform a solution into a solution. Later, Meleshko and Moyo [8], applying this generalized theory and extended group classification notion to treat delay differential equations, presented the complete group classification of the reaction–diffusion equation with discrete delay. Pue-on et al. [11] applied this generalized theory to second-order delay ordinary differential equations $y'' = f(x,\ y,\ y_\tau,\ y',\ y'_\tau)$, where $y_\tau = y(x-\tau),\ y'_\tau = y'(x-\tau)$ and provided all classes of second-order delay ordinary differential equations admitting a Lie algebra. Recently, Tanthanuch [15] provided a new application of this theory and give a complete group classification of the nonhomogeneous inviscid Burgers equation with delay.

We find that the study of delay differential equations by group approach focuses on discrete delay and the main method is group analysis theory. In this paper, we will discuss the symmetry and some corresponding group-invariant solutions of reaction–diffusion equation with distributed delay, which has the following form

$$u_t(x,t) = u_{xx}(x,t) + F(t,x,u(x,t),(f*u)(x,t)), \tag{1.1}$$

where u is the function of t, x, with $x \in \mathbb{R}$ being the spatial variable, t is the time variable. And the convolution $f*u$ is defined by

$$(f*u)(x,t) = \int_{-\infty}^{t} f(t-s)u(x,s)ds,$$

where the kernel $f : [0,\infty) \to [0,\infty)$ satisfies the following normalization assumption:

$$f(t) \geqslant 0, \quad \text{for all} \quad t \geqslant 0, \quad \int_0^\infty f(t)dt = 1. \tag{1.2}$$

Two specific cases of delay kernel function $f(t)$, which have been widely used, are weak generic kernel [1]:

$$f(t) = \frac{1}{\tau}e^{-t/\tau}, \quad \tau > 0 \tag{1.3}$$

and strong generic kernel:

$$f(t) = \frac{t}{\tau^2}e^{-t/\tau}, \quad \tau > 0. \tag{1.4}$$

Here, we assume that $f(t)$ is weak generic kernel. the parameter τ measures the delay, (1.3) implies that the importance of events in the past simply decreases exponentially the further one looks into the past.

The existence of solutions of (1.1) with (1.2) can be found in [12,18]. It is significant to study the symmetries of system (1.1) with (1.2). Utilizing the generalized symmetries theory and undetermined coefficient method of differential equation, we mainly study the symmetry and some corresponding group-invariant solutions of the reaction–diffusion equation with weak distributed delay in this paper. The research results show that different cases of reactions, $F(t,x,u,v)$, possess different symmetries and group-invariant solutions, and we show several common cases of reactions' symmetry and group-invariant solution.

The rest of this chapter is organized as follows. Section 2 gives the theory of generalized symmetries. Section 3 discusses the characteristic of generalized symmetries of reaction–diffusion equations with distributed delay. In the process of solving the equation, only those relatively simple symmetries are useful. We determine some of group-invariant solutions, corresponding to the symmetry of reaction–diffusion equation with distributed delay in Section 4.

2. Generalized symmetry method

In this section, we investigate the theory of generalized symmetry [9,16].

It is useful to introduce some notations. Suppose $(x,\ t) \in \mathbb{R}^2$ is independent variable and $u = (u^1, u^2) \in \mathbb{R}^2$ is dependent variable. $\mathcal{A}$ denotes the space of smooth differential functions $P(x,t,u^{(n)})$, depending on $x,\ t,\ u$ and derivatives of u up to some finite, but unspecified order n, the range of P is $\mathbb{R}$. Let $P[u] := P(x,t,u^{(n)})$.

Definition 2.1. A *generalized vector field* has the form

$$\mathbf{v} = \xi[u]\frac{\partial}{\partial x} + \eta[u]\frac{\partial}{\partial t} + \phi_1[u]\frac{\partial}{\partial u^1} + \phi_2[u]\frac{\partial}{\partial u^2}, \tag{2.1}$$