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Anomalous diffusion approach to non-exponential relaxation in complex physical systems

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ABSTRACT

We derive the relaxation function from the simple model of two-state systems under memory effects caused by the subordination. The non-exponential relaxation is shown to result from subordination by inverse infinity divisible random processes. The wide class of such random processes includes ordinary α-stable, tempered α-stable, exponential, gamma processes and many others as particular cases. This approach generalizes the Cole–Cole, Cole–Davidson and Havriliak–Negami laws well known in experimental physics of relaxation. The presented considerations discover a direct (one-to-one) relationship between the method of random relaxation rates and the anomalous diffusion approach based on subordination of random processes that are applied for the theory of relaxation phenomena. Moreover, it is found that the space and time clusterizations are responsible on equal foots for power-law memory effects in relaxation of complex physical systems.

1. Introduction

The major and interesting feature of relaxation in many complex systems is their non-exponential nature [1–4]. Consisting of many components, the complex dynamics of relaxing systems is that the components interact with each other, and often they cannot tend to equilibrium independently. Evolution of their initially imposed states has a random character, although in a whole each relaxing system shows a deterministic macroscopic response. The response is just a gleam of the random processes responsible for the complex dynamics. A natural technique, accounting local random characteristics of complex systems for the derivation of deterministic and universally valid empirical relaxation laws, is the general probabilistic formalism of limit theorems, regardless of the precise nature of local interactions in the systems. However, the classical methodology of relaxation, based on random values with finite expected values, breaks down in complex systems because of their scaling properties. Really, wide-range experimental data demonstrate fractional power laws for the complex susceptibility behavior at low or/and high frequencies. To interpret such results, the probability distributions for evaluating the dynamical averages of relaxation processes should have infinite variance. That is why a special place in the theory of relaxation is occupied by random processes with long-term tails in probability distributions.

The purpose of this paper is to shed light on the role of infinitely divisible random processes in the macroscopic character of relaxation processes in disordered materials. To reach this goal we present a general scenario of subordination based on

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random processes that extend the theory of non-exponential relaxation [5]. The basic idea presented here also starts with subordination of random processes independent of each other. Long-tailed random processes as subordinators lead to power-law memory effects in such dynamical system. As a result, the same asymptotic properties are inherent in relaxation functions. The derivation of memory function for different relaxation processes is of interest, as it gives many advantages in clarifying the nature of relaxation phenomena.

The paper is organized as follows: at first, we briefly define the primary random processes and formulate a modification of their subordination representation. Next, within this approach we derive the frequency-domain relaxation function in an analytical form in Section 3. The general scenario permits ones to analyze a wider range of relaxation processes in Sections 4 and 5. In the main Section 6 we show how the results of our study can be applied to experimentally observed cases of the relaxation phenomena. Moreover, it is found that the space and time clusterizations are responsible on equal foots for power-law memory effects in relaxation of complex physical systems (Fig. 3). In the last Section the conclusions are drawn.

2. Infinitely divisible processes as subordinators

Let us start with some basic facts concerning infinitely divisible distributions and the corresponding random processes. Distribution of a nonnegative random variable T is infinitely divisible, if its Laplace transform takes the form

$$\langle e^{-sT(\tau)}\rangle = e^{-\tau\Psi(s)}, \tag{1}$$

where $\Psi(s)$ is the Laplace exponent according to the Lévy–Khintchine formula [6]. The function $\Psi(s)$ is also called the Lévy exponent and can be written as

$$\Psi(s) = \lambda s + \int_0^\infty (1 - e^{-sx})\mu(dx),$$

where λ is the drift parameter, $\mu(dx)$ the appropriate Lévy measure. For a well-defined random process with an infinitely divisible distribution, the function $\Psi(s)$ should be nonnegative with a complete monotone first derivative and $\Psi(0) = 0$ [6]. Infinitely divisible distributions were firstly introduced by de Finetti in 1929 [7] and have been studied fruitfully by Kolmogorov, Lévy and Khintchine later [8]. The class of functions to which belongs $\Psi(s)$ is named Bernstein functions [9]. Infinitely divisible random processes are very appropriate to be subordinators.

Recall that subordination of the random process $X(\tau)$ by another process $S(t)$ means a randomization of physical time t by $S(t)$. Thus, a new process $Y(t) = X[S(t)]$ consists of two random processes such that one of them, $X(\tau)$, is a parent, and $S(t)$ determines a new operational time τ. The operational time $S(t)$ as an inverse process to a non-decreasing Lévy process $T(\tau)$ fulfills the following first passage time relation

$$S(t) = \inf\{\tau > 0 : T(\tau) > t\}.$$

Since Lévy processes have infinitely divisible distributions, one can define this subordinator through a sum of independent and identically distributed (iid) random variables with an infinitely divisible probability density function (pdf). There exist many examples of such pdfs. Among them are well-known such as Gaussian, inverse Gaussian, α-stable, tempered α-stable, exponential, gamma, compound Poisson, Pareto, Linnik, Mittag–Leffler and others, including right-skewed α-stable distributions [6]. If $f(t,\tau)$ is the pdf of $T(\tau)$, then the mean $\langle e^{-sT(\tau)}\rangle$ is the Laplace transform of $f(t,\tau)$ equal to

$$\langle e^{-sT(\tau)}\rangle = \int_0^\infty e^{-st} f(t,\tau)dt.$$

Knowing $f(t,\tau)$, it is not difficult to find the pdf $g(\tau,t)$ of its inverse subordinator $S(t)$ as

$$g(\tau,t) = -\frac{\partial}{\partial\tau}\int_{-\infty}^{t} f(t',\tau)dt'.$$

Based on the Laplace transform of $g(\tau,t)$ with respect to t we come to

$$\tilde{g}(\tau,s) = \frac{\Psi(s)}{s} e^{-\tau\Psi(s)}. \tag{2}$$

It should be pointed out that the exponential form of the Laplace image $\tilde{g}(\tau,s)$ allows ones to simplify further calculations by reducing them to algebraic transformations.

3. Relaxation in two-state systems

The simplest interpretation of relaxation phenomena assumes a dynamical system of independent exponentially relaxing components (for example, dipoles) with different (but independent) relaxation rates [10]. Following this law (called Debye's), such a relaxation process may be described in the form of a two-state system. If N is the common number of dipoles in a dielectric system, then $N_\uparrow$ is the number of dipoles in the state $\uparrow$, and $N_\downarrow$ is the number of dipoles in the state $\downarrow$. Their sum