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Analytical studies of a time-fractional porous medium equation. Derivation, approximation and applications

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ABSTRACT

In this paper we investigate the porous medium equation with a time-fractional derivative. We justify that the resulting equation emerges when we consider a waiting-time (or trapping) phenomenon that can have its place in the medium. Our deterministic derivation is dual to the stochastic CTRW framework and can include nonlinear effects. With the use of the previously developed method we approximate the investigated equation along with a constant flux boundary conditions and obtain a very accurate solution. Moreover, we generalise the approximation method and provide explicit formulas which can be readily used in applications. The subdiffusive anomalies in some porous media such as construction materials have been recently verified by experiment. Our simple approximate solution of the time-fractional porous medium equation fits accurately a sample data which comes from one of these experiments.

1. Introduction

Throughout the last decade the number of experimental reports of the anomalous diffusive phenomena has grown. For example, some of these concern moisture dispersion in building materials (see for ex. [1–5]) but also some other minerals like zeolite [6,7]. When we consider an essentially one dimensional (other dimensions can be neglected) porous medium with one boundary kept at constant moisture, the concentration in subsequent time instant t and space point x is a function of $x/\sqrt{t}$. This characteristic space–time scaling is not, however, present in every type of porous media. As experiments showed, in certain materials the moisture propagates according to the $x/t^{\alpha/2}$ scaling, where $0 < \alpha < 2$. When $0 < \alpha < 1$ the moisture concentration represents a subdiffusive character, meaning that the fluid particles can be trapped in some regions for a significant amount of time. In the superdiffusive case $1 < \alpha < 2$ there is a possibility that water can be transported for large distances in a relatively short time (jump behaviour). All of these phenomena are certainly associated with the geometrical aspects of the medium such as pore distribution or heterogeneous features such as highly conductive channels and macropores [8]. Some Authors state that the anomalous behaviour can also be caused by chemical reactions that force the diffusivity to change during the imbibition [9].

There have been many attempts to model the anomalous diffusion in various porous media such as building materials. For example in [5] a model based on a nonlinear Fickian Law has been proposed. The most popular approach, however, was made by the use of the fractional derivatives [10,11,8,12]. These generalizations of the classical derivative operator are necessarily nonlocal. This gives an opportunity to model the history (nonlocality in time) or long-range influence in medium (nonlocality in space). The theoretical foundation stating that the transport in porous media should include nonlocal

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phenomena was developed in [13–15], where Authors devised the nonlocal Darcy's Law from the Boltzmann transport equation and showed that the nonlocality can be a consequence of the heterogeneous pore distribution. Due to utmost importance for industry and agriculture, various kinds of porous media are in the centre of attention of a number of researchers. Results concerning a flow of non-Newtonian fluids exhibit very nontrivial results in applied mathematics (for a treatment of variable viscosity see for ex. [16–19]). Some other industrially-inspired investigations concern a peristaltic flow through various ducts [20,21]. Moreover, a phenomenon of porosity redistribution and swelling medium was also observed very recently in the case of building materials [22], which are modelled in our present paper. Some porous media show very peculiar phenomena such as second consolidation. It is interesting that these can also be successfully modelled by the fractional derivative (see [23]). Fractional derivatives arise naturally where the notion of nonlocality is a main concept of the transport process. The fractional derivative approach has an advantage to offer a theoretical framework in which all the anomalous behaviours can be governed by the derivative order. The main disadvantage, however, is that the fractional equations are always more difficult to solve explicitly (either analytically or numerically) especially in the nonlinear case.

In this work we generalise our previous results concerning the time-fractional porous medium equation, which is used to model anomalous moisture dispersion. The initial results were published in [24] and further developed in [25]. In what follows we present a theoretical derivation of the time-fractional anomalous diffusion equation and provide an analysis of its approximate solutions. The boundary conditions, that we supplement, are of the two types: constant concentration or constant flux at the interface. Both of these have a direct physical interpretation. We present our approximation method and show some estimates on the error terms. All of our theoretical considerations are illustrated by numerical analysis and fitting with real experimental data. Both of these verify that our approximations are sensibly accurate when describing anomalous diffusion in porous media. Moreover, a simple and closed form of the formulas we derive is an additional advantage which can quicken the data fitting process in applications. The usual way of data fitting is being done with a numerical solution of the nonlinear anomalous diffusion equation. Due to the nonlocality of the fractional operator, the computational power needed for the nonlinear finite difference method is very large. Simple (but approximate) solutions, which we derive, can be a remedy for these computational problems. Other works found in the literature treat the governing equation only numerically. We think that our contribution will enhance the performance and understanding of those numerical calculations.

2. Model of the time-fractional anomalous diffusion

2.1. Formulation

Consider a porous medium in which the Darcy's Law holds [26]

$$q = -\frac{\rho k}{\mu}\nabla p, \tag{1}$$

where q is the flux of the fluid with viscosity μ flowing under the pressure p through the medium of permeability k. In any porous medium not all of its space can be filled with the flow. The *concentration* $u = u(x,t)$ is defined to be density of the water inside the Representative Elementary Volume (REV) at position x and time instant t. The total fraction of space available for the fluid is called the *porosity* of the medium. Under suitable assumptions [27] the pressure p can be expressed as a monotone function of the concentration u, so we can write

$$q = -\frac{\rho k}{\mu}\frac{dp}{du}\nabla u =: -D(u)\nabla u, \tag{2}$$

where D denotes the diffusivity, which can depend on the moisture concentration u. In applications this is almost always the case [27] – the diffusivity can change even a several orders of magnitude with the change of u. In any region of the porous medium the amount of fluid must be conserved and thus the moisture must obey the continuity equation

$$u_t + \nabla\cdot q = 0, \tag{3}$$

where subscript denotes the derivative with respect to time variable t. Above equation along with (2) gives us the governing nonlinear diffusion equation known in the hydrology as the *Richards equation* [28]

$$u_t = \nabla\cdot(D(u)\nabla u), \tag{4}$$

If we included the gravity we would obtain an additional convective term. But in our model it is sufficient to consider only the diffusive phenomena. For our purposes it is also sufficient to reduce the equation to describe one spatial dimension only since in the experiment we want to model, the diffusion progresses mostly in one direction. In this work we will consider two different initial-boundary conditions for (4), namely

$$u(0,t) = C, \quad u(x,0) = 0, \quad x > 0, \quad t > 0, \tag{5}$$

which models a constant concentration at the face of an initially dry sample and

$$-D(u(0,t))u_x(0,t) = Q, \quad u(x,0) = 0, \quad x > 0, \quad t > 0, \tag{6}$$