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A note on shadowing with chain transitivity

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ABSTRACT

Let $f : X \to X$ be a continuous map of a compact metric space X. The map f induces in a natural way a map f_M on the space $M(X)$ of probability measures on X, and a transformation f_K on the space $K(X)$ of closed subsets of X. In this paper, we show that if (X,f) is a chain transitive system with shadowing property, then exactly one of the following two statements holds:

(a) f^n and $(f_K)^n$ are syndetically sensitive for all $n \geqslant 1$.
(b) f^n and $(f_K)^n$ are equicontinuous for all $n \geqslant 1$.

In particular, we show that for a continuous map $f : X \to X$ of a compact metric space X with infinite elements, if f is a chain transitive map with the shadowing property, then f^n and $(f_K)^n$ are syndetically sensitive for all $n \geqslant 1$. Also, we show that if f_M (resp. f_K) is chain transitive and syndetically sensitive, and f_M (resp. f_K) has the shadowing property, then f is sensitive.

In addition, we introduce the notion of ergodical sensitivity and present a sufficient condition for a chain transitive system (X,f) (resp. $(M(X),f_M)$) to be ergodically sensitive. As an application, we show that for a $\mathcal{L}$-hyperbolic homeomorphism f of a compact metric space X, if f has the AASP, then f^n is syndetically sensitive and multi-sensitive for all $n \geqslant 1$.

1. Introduction

It is well known that sensitive dependence characterizes the unpredictability of chaotic phenomenon. The dependence is the essential condition of various definitions of a system to be chaotic. Therefore, when does a system have sensitive dependence? This question has gained some attention in more recent papers (see [1–13]). Throughout the paper, by a dynamical system we mean a pair (X,f), where X is a nontrivial and compact metric space with metric d and $f : X \to X$ is a continuous map.

Roughly speaking, a dynamical system (X,f) is sensitive if for any region U of the phase space, there exist two points in U and an integer $n \geqslant 0$ such that the nth iterates of the two points under the map f are significantly separated. The largeness of the set of all $n \in Z^+$ where this significant separation or sensitivity happens can be thought of as a measure of how sensitive the dynamical system is. Especially, if this set is quite thin with arbitrarily large gaps between consecutive entries, then one has some excuse for treating the dynamical system as practically non-sensitive!

For continuous self-maps of compact metric spaces, Moothathu [11] initiated a preliminary study of stronger forms of sensitivity formulated in terms of large subsets of Z^+. Mainly he considered syndetic sensitivity and cofinite sensitivity. Also, he constructed a transitive, sensitive map which is not syndetically sensitive and obtained the following result.

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Theorem 1.1. *(1) Any syndetically transitive, non-minimal map is syndetically sensitive(this improves the result that sensitivity is redundant in Devaney's definition of chaos). (2) Any sensitive map of [0,1] is cofinitely sensitive. (3) Any sensitive subshift of finite type is cofinitely sensitive. (4) Any syndetically transitive, infinite subshift is syndetically sensitive. (5) No Sturmian subshift is cofinitely sensitive.*

Let $f : X \to X$ be a continuous map of a compact metric space X. The map f induces in a natural way a map f_M on the space $M(X)$ of probability measures on X, and a transformation f_K on the space $K(X)$ of closed subsets of X. In this paper, we show that if (X,f) is a chain transitive system with shadowing property, then f^n and $(f_K)^n$ are syndetically sensitive for all $n \geqslant 1$ or f^n and $(f_K)^n$ are equicontinuous for all $n \geqslant 1$. In particular, we show that for a continuous map $f : X \to X$ of a compact metric space X with infinite elements, if f is a chain transitive map with the shadowing property, then f^n and $(f_K)^n$ are syndetically sensitive for all $n \geqslant 1$. Also, it is proven that if f_M (resp. f_K) is a chain transitive map with the shadowing property, and f_M (resp. f_K) is syndetically sensitive, then f is sensitive.

In addition, we introduce the notion of ergodical sensitivity and present a sufficient condition for a chain transitive system (X,f) (resp. $(M(X),f_M)$) to be ergodically sensitive. As an application, we show that for a $\mathcal{L}$-hyperbolic homeomorphism f of a compact metric space X, if f has the AASP, then f^n is syndetically sensitive and multi-sensitive for all $n \geqslant 1$.

Our results improve and extend the existing results. The organization of this paper is as follows. In Section 2, we recall some concepts and useful lemmas. Main results are established in Section 3.

2. Preliminaries

Firstly we complete some notations and recall some concepts.

We shall use $|A|$ to denote the cardinality of A. Let (X,f) be a dynamical system. For any two nonempty sets $U, V \subset X$, we write $N_f(U,V) = \{n \in Z^+ : U \bigcap f^{-n}(V) \neq \emptyset\}$. Obviously, we have $N_f(U,V) = \{n \in Z^+ : f^n(U) \bigcap V \neq \emptyset\}$.

A subset $S \subset Z^+$ is called the positive upper density if

$$\limsup_{k\to\infty} \frac{1}{k+1} |\{0 \leqslant j \leqslant k : j \in S\}| > 0.$$

A subset $S \subset Z^+$ is thick if S contains arbitrarily large blocks of consecutive numbers.

A map $f : X \to X$ is topologically transitive if $N_f(U,V)$ is nonempty, for any nonempty open sets $U, V \subset X$.

A map $f : X \to X$ is topologically ergodic if $N_f(U,V)$ has positive upper density, for any nonempty open sets $U, V \subset X$.

A map $f : X \to X$ is topologically weak mixing if $f \times f$ is topologically transitive.

A map $f : X \to X$ is syndetically transitive if $N_f(U,V)$ is syndetic, that is, $Z^+ \backslash N_f(V,\delta)$ is not thick, for any nonempty open sets $U, V \subset X$.

Let (X,f) be a dynamical system. According to the classical definition, f has sensitive dependence if there is $\delta > 0$ such that for any $x \in X$ and any open neighborhood V_x of x, there is $n \in Z^+$ such that $\sup\{d(f^n(x),f^n(y)) : y \in V_x\} > \delta$. We can write this in a slightly different way. For $V \subset X$ and $\delta > 0$, let $N_f(V,\delta) = \{n \in Z^+$: there exist $x, y \in V$ with $d(f^n(x),f^n(y)) > \delta\}$. Now, we say:

(1) f is sensitive if there is $\delta > 0$ such that for any nonempty open set $V \subset X$, $N_f(V,\delta)$ is nonempty.
(2) f is syndetically sensitive if there is $\delta > 0$ such that for every nonempty open subset $V \subset X$, $N_f(V,\delta)$ is syndetic.
(3) f is ergodically sensitive if there is $\delta > 0$ such that for every nonempty open subset $V \subset X$, $N_f(V,\delta)$ has positive upper density.
(4) f is multi-sensitive if there is $\delta > 0$ such that for every integer $k > 0$ and for any nonempty open subsets $V_1, V_1, \ldots, V_k \subset X, \bigcap_{i=1}^k N_f(V_i,\delta) \neq \emptyset$.

It is known from [11] that multi-sensitive and syndetically sensitive are stronger than sensitive. Clearly, syndetical sensitivity implies ergodical sensitivity.

In here and the following, the $\sigma -$ algebra of Borel subsets of a compact metric space X will be denoted by $\mathcal{B}(X)$. We will denote by $M(X)$ the collection of all probability measures defined on the measurable space $(X, \mathcal{B}(X))$. We call the members of $M(X)$ Borel probability measures on X. Each $x \in X$ determines a member δ_x (that is, point measure) of $M(X)$ defined by $\delta_x(A) = 1$ if $x \in A$; $\delta_x(A) = 0$ if $x \notin A$. So, the map $x \to \delta_x$ imbeds X inside $M(X)$. Let (X,f) be a dynamical system. It is well known that the map defined by $f_M(\mu)(B) = \mu(f^{-1}(B))$ for any $\mu \in M(X)$ and any $B \in \mathcal{B}(X)$ and the map $x \to \delta_x$ from X into $M(X)$ are continuous, and $M(X)$ is a nonempty convex set which is compact in the weak topology (see [14,15]). Clearly, the map $x \to \delta_x$ imbeds X inside $M(X)$. It is well known that the convex combinations of point measures (i. e. the measures with finite support) are dense in $M(X)$ (see [14,15]).

Assume that X is a compact metric space with metric d and $M(X)$ is the space of Borel probability measures on X provided with the Prohorov metric p defined by $p(\lambda,\mu) = \inf\{\varepsilon : \lambda(A) \leqslant \mu(A^\varepsilon) + \varepsilon$ and $\mu(A) \leqslant \lambda(A^\varepsilon) + \varepsilon$ for all Borel sets $A \in \mathcal{B}(X)\}$ for λ, $\mu \in M(X)$, where $A^\varepsilon = \{xeX : d(x,A) < \varepsilon\}$. As Stassen showed in [16], we have $p(\lambda,\mu) = \inf\{\varepsilon : \lambda(A) \leqslant \mu(A^\varepsilon) + \varepsilon$ for all Borel sets $A \in \mathcal{B}(X)\}$. The induced topology is just the weak topology [14,15] for measures. It turns $M(X)$ into a compact space [14,17].

Let X be a compact metric space with metric d and $K(X)$ be the space of closed subsets of X provided with the Hausdorff metric h defined by