Commun Nonlinear Sci Numer Simulat 17 (2012) 3050–3060

Contents lists available at SciVerse ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

On the concept and existence of solution for impulsive fractional differential equations ☆

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ARTICLE INFO

Article history:
Received 24 July 2011
Received in revised form 30 September 2011
Accepted 14 November 2011
Available online 25 November 2011

Keywords:
Caputo fractional derivative
Impulsive differential equations
Existence

ABSTRACT

This paper is motivated from some recent papers treating the problem of the existence of a solution for impulsive differential equations with fractional derivative. We firstly show that the formula of solutions in cited papers are incorrect. Secondly, we reconsider a class of impulsive fractional differential equations and introduce a correct formula of solutions for a impulsive Cauchy problem with Caputo fractional derivative. Further, some sufficient conditions for existence of the solutions are established by applying fixed point methods. Some examples are given to illustrate the results.

1. Introduction

Fractional differential equations have recently proved to be strong tools in the modeling of many physical phenomena. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more details on fractional calculus theory, one can see the monographs of Diethelm [1], Kilbas et al. [2], Lakshmikantham et al. [3], Miller and Ross [4], Michalski [5], Podlubny [6] and Tarasov [7]. Fractional differential equations involving the Riemann–Liouville fractional derivative or the Caputo fractional derivative have been paid more and more attentions (see for example [8–18]).

This paper is motivated from some recent papers treating the problem of the existence of solutions for impulsive differential equations with fractional derivative. By directly computation it is easy to see that the concepts of piecewise continuous solutions used in many papers are not appropriate. To prove our claim, consider the impulsive problem

$$\begin{cases} {}^cD^q_{0,t}u(t) := {}^cD^q_t u(t) = f(t,u(t)), t \in J' := J \setminus \{t_1,\ldots,t_m\}, \quad J := [0,T], \\ \Delta u(t_k) = I_k\big(u(t_k^-)\big), \quad k = 1,2,\ldots,m, \\ u(0) = u_0, \end{cases} \tag{1}$$

where ${}^cD^q_t$ is the Caputo fractional derivative of order $q \in (0,1)$ with the lower limit zero, $u_0 \in R$, $f : J \times R \to R$ is jointly continuous, $I_k : R \to R$ and t_k satisfy $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$.

☆ The first author was partially supported by Grants VEGA-MS 1/0507/11 and APVV-0134-10; the second author was supported by National Natural Science Foundation of China (10971173); the third author was supported by Key Projects of Science and Technology Research in the Chinese Ministry of Education (211169), Tianyuan Special Funds of the National Natural Science Foundation of China (11026102).

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doi:10.1016/j.cnsns.2011.11.017

The approach used in the cited papers [19–23] is based on the results of Eq. (1) is equivalent to the following integral equation

$$u(t)=\begin{cases} u_0+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s))ds, & \text{for } t\in[0,t_1),\\ \vdots \\ u_0+\frac{1}{\Gamma(q)}\sum\limits_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}f(s,u(s))ds+\frac{1}{\Gamma(q)}\int_{t_k}^t(t-s)^{q-1}f(s,u(s))ds+\sum\limits_{0<t_k<t}I_k(u(t_k^-)), & \text{for } t\in(t_k,t_{k+1}],\quad k=1,2,\dots m. \end{cases} \tag{2}$$

Then one can say that a function $u\in PC^1(J,R)$ is called a solution of Eq. (1) if u satisfies Eq. (2). However, it is easy to see that this concept of a solution is not realistic. Let's consider the following counterexample:

$$\begin{cases} {}^cD_t^{\frac{1}{4}}u(t)=t, & t\in(0,2]\setminus\{1\},\\ u(0)=0,\\ u(1^+)=u(1^-)+1. \end{cases} \tag{3}$$

As a special case of Eq. (1), the solution of Eq. (3) is given by

$$u(t)=\frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_0^1(1-s)^{\frac{1}{4}-1}s\,ds+\frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_1^t(t-s)^{\frac{1}{4}-1}s\,ds+1,\quad \text{for}\quad t\in(1,2]. \tag{4}$$

Then one can verify that u given by (4) does not satisfy (3). In fact, since the general solution of the first equation of (3) is given by $u(t)=c+\frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_0^t(t-s)^{\frac{1}{4}-1}sds=c+\frac{16}{5\Gamma\left(\frac{1}{4}\right)}t^{\frac{5}{4}}$ on any open subinterval of $(0,2]\setminus\{1\}$ for a constant c, it is easy to see that

$$u(t)=\begin{cases} \frac{16}{5\Gamma\left(\frac{1}{4}\right)}t^{\frac{5}{4}} & \text{for } 0\leqslant t<1,\\ 1+\frac{16}{5\Gamma\left(\frac{1}{4}\right)}t^{\frac{5}{4}} & \text{for } 1<t\leqslant 2 \end{cases} \tag{5}$$

solves (3). On the other hand, formula (4) gives

$$u(t)=\frac{16}{5\Gamma\left(\frac{1}{4}\right)}+1+\frac{4}{5\Gamma\left(\frac{1}{4}\right)}(t-1)^{\frac{1}{4}}(4t+1)$$

for $1<t\leqslant 2$, which is different from (5). After making some carefully analysis, one can find that the source of the error should be the memory property of fractional calculus. Many researchers consider the problem on interval $(1,2]$ with the Caputo derivative ${}^cD_{1,t}^{\frac{1}{4}}$ and not ${}^cD_t^{\frac{1}{4}}$. Unfortunately, it does not hold that ${}^cD_t^{\frac{1}{4}}$ restricted on interval $(1,2]$ is ${}^cD_{1,t}^{\frac{1}{4}}$.

Motivated by the above remarks, we reconsider the impulsive differential equations with Caputo fractional derivative and seek a correct formula of the solution for this problem. Consider the Cauchy problems for the following impulsive fractional differential equations

$$\begin{cases} {}^cD_t^qu(t)=f(t,u(t)), & t\in J',\\ u(t_k^+)=u(t_k^-)+y_k, & k=1,2,\dots,m,\\ u(0)=u_0, & y_k\in R. \end{cases} \tag{6}$$

In the present paper, we try to seek a correct formula of solutions for Eq. (6). Fortunately, we find that the formula of solutions for Eq. (6) should be

$$u(t)=\begin{cases} u_0+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s))ds, & \text{for}\quad t\in[0,t_1),\\ u_0+y_1+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s))ds, & \text{for}\quad t\in(t_1,t_2),\\ u_0+y_1+y_2+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s))ds, & \text{for}\quad t\in(t_2,t_3),\\ \vdots \\ u_0+\sum\limits_{i=1}^m y_i+\frac{1}{\Gamma(q)}\int_0^t(t-s)^{q-1}f(s,u(s))ds, & \text{for}\quad t\in(t_m,T]. \end{cases} \tag{7}$$

The rest of this paper is organized as follows. In Section 2, we give some notations, recall some concepts and preparation results, and introduce a concept of a piecewise continuous solution for our problem. In Section 3, we give three main results, the first result based on Banach contraction principle, the second result based on Schaefer's fixed point theorem, the third result based on nonlinear alternative of Leray–Schauder type. Some examples are given in Section 5 to demonstrate the application of our main results.