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Primary resonance of Duffing oscillator with fractional-order derivative

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ABSTRACT

In this paper the primary resonance of Duffing oscillator with fractional-order derivative is researched by the averaging method. At first the approximately analytical solution and the amplitude–frequency equation are obtained. Additionally, the effect of the fractional-order derivative on the system dynamics is analyzed, and it is found that the fractional-order derivative could affect not only the viscous damping, but also the linear stiffness, which is characterized by the equivalent damping coefficient and the equivalent stiffness coefficient. This conclusion is remarkably different from the existing research results about nonlinear system with fractional-order derivative. Moreover, the comparisons of the amplitude–frequency curves by the approximately analytical solution and the numerical integration are fulfilled, and the results certify the correctness and satisfactory precision of the approximately analytical solution. At last, the effects of the two parameters of the fractional-order derivative, i.e. the fractional coefficient and the fractional order, on the amplitude–frequency curves are investigated, which are different from the traditional integer-order Duffing oscillator.

1. Introduction

Fractional-order derivative and integral was presented in the late 1700s, and a lot of investigations, both on the general theory and application of fractional-order derivative had been issued by many authors [1–28] since then. The theoretical research was focused on the definition, properties, and computation method of the fractional-order derivative and integral. In the application, two aspects were important, such as the description of the memory and hereditary properties in various materials and processes, and artificial introduce of the fractional-order feedback in the control engineering. Moreover, the effects of the fractional-order derivative on dynamical system were interesting and meaningful, and many issued works were fulfilled on this subject.

Works on system dynamics with fractional-order derivative may be divided into several groups, one of which is the qualitatively analysis on the number and stability of solutions. For example, Machado and Galhano [10] analyzed statistical dynamics of a large number of micromechanical masses, and found the existence of both integer and fractional properties in the global dynamics. Li et al. [11] studied the stable parameters range of the simplified Mathieu-type equation with fractional-order derivative. By using the idea of stability switch, Wang and Hu [12] and Wang and Du [13] investigated a linear single degree-of-freedom (SDOF) oscillator with fractional-order derivative, and found some important phenomena. Rossikhin and Shitikova [14] proposed a method to analyze the free damped vibrations of a fractional-order oscillator. Tavazoei et al. [15], Pinto and Machado et al. [16] studied the fractional-order van der Pol oscillator and found multiple limit cycles existing in the system.

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Due to the complexity of fractional-order derivative, numerically investigation on the complicated nonlinear dynamics phenomena such as bifurcation, chaos and synchronization were another important group in the dynamical system with fractional-order derivative. Atanackovic and Stankovic [17] proposed a modified numerical procedure to solve fractional-order differential equations, and the test on several examples verified the efficiency of the method. Cao et al. [18] simulated the fractional-order Duffing equation and investigated the effect of the fractional order on system dynamics using phase diagram, bifurcation diagram and Poincaré map. Sheu et al. [19] solved the fractional damped Duffing equation by transforming them into a set of fractional integral equations. Wu et al. [20], Chen and Chen [21] and Lu [22] studied the synchronization in fractional-order nonlinear system.

Analytical research was also important in dynamical system, and there were some important works about analytical investigation on dynamical system with the fractional-order derivative. Qi and Xu [23] analyzed the unsteady flow of viscoelastic fluid with the fractional-order derivative Maxwell model. Wahi and Chatterjee [24] studied an oscillator with fractional-order derivative and time-delay. Chen and Zhu [25], Padovan and Sawicki [26], Borowiec et al. [27], Huang and Jin [28] also investigated different fractional-order system and presented important results by analytical research. However, the analytical researches were focused on some special fractional orders, or the fractional-order derivative was simply considered as the special damping force, which may be insufficient in some cases.

In this paper, the Duffing oscillator with fractional-order derivative is researched analytically. In Section 2 the primary resonance of the Duffing oscillator with fractional-order derivative is investigated, where two important formulae are given and the approximately analytical solution is obtained. Additionally, the effects of the fractional-order derivative on the damping and stiffness are formulated as the equivalent damping coefficient and the equivalent stiffness coefficient, which is remarkably different from the results in most other works. Section 3 presents the steady-state solution, the amplitude–frequency equation, and the stability condition of the steady-state solution. At last, the comparison of the approximately analytical solution with the numerical result is fulfilled in Section 4, and the effects of the parameters for the fractional-order derivative on the amplitude–frequency equation are also given in this section.

2. The approximately analytical solution of Duffing oscillator with fractional-order derivative

The considered Duffing oscillator with fractional-order derivative is

$$m\ddot{x}(t) + kx(t) + c\dot{x}(t) + \alpha_1 x^3(t) + K_1 D^p[x(t)] = F\cos(\omega t), \tag{1}$$

where m, k, c, α_1, F, ω are the system mass, linear stiffness coefficient, linear viscous damping coefficient, nonlinear stiffness coefficient, excitation amplitude and excitation frequency respectively, and $D^p[x(t)]$ is the p-order derivative of $x(t)$ to t with the fractional coefficient $K_1 (K_1 > 0)$ and the fractional order $p (0 \leqslant p \leqslant 1)$. There are several definitions for fractional-order derivative, and they are equivalent under some conditions for a wide class of functions. Here we adopt Caputo's definition

$$D^p[x(t)] = \frac{1}{\Gamma(1-p)} \int_0^t \frac{x\prime(u)}{(t-u)^p} du, \tag{2}$$

where $\Gamma(z)$ is Gamma function satisfying $\Gamma(z+1) = z\Gamma(z)$.

Using the following transformation of coordinates

$$\omega_0 = \sqrt{\frac{k}{m}}, \quad 2\varepsilon\mu = \frac{c}{m}, \quad \varepsilon\alpha = \frac{\alpha_1}{m}, \quad \varepsilon k_1 = \frac{K_1}{m}, \quad \varepsilon f = \frac{F}{m},$$

Eq. (1) becomes

$$\ddot{x}(t) + \omega_0^2 x(t) + 2\varepsilon\mu\dot{x}(t) + \varepsilon\alpha x^3(t) + \varepsilon k_1 D^p[x(t)] = \varepsilon f\cos(\omega t), \tag{3}$$

where ω_0 is natural frequency. In this transformation, ε, μ, α, k_1 and f are not dimensionless quantity, and the transformation is only to satisfy the requirement of averaging method formally. The primary resonance means the excitation frequency is close to the natural one, i.e. $\omega \approx \omega_0$. In order to illustrate the approximate degree, one should introduce

$$\omega^2 = \omega_0^2 + \varepsilon\sigma, \tag{4}$$

where σ is the detuning factor. Eq. (3) could be transformed into

$$\ddot{x}(t) + \omega^2 x(t) = \varepsilon\{f\cos(\omega t) + \sigma x(t) - 2\mu\dot{x}(t) - \alpha x^3(t) - k_1 D^p[x(t)]\}. \tag{5}$$

Supposing Eq. (5) has the solution as

$$x = a\cos\varphi \tag{6a}$$

and

$$\dot{x} = -a\omega\sin\varphi, \tag{6b}$$

where the amplitude a and the generalized phase φ ($\varphi = \omega t + \theta$) are slow varying functions of t. By differentiating Eq. (6a) to t, one could obtain