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# Lagrangian descriptors: A method for revealing phase space structures of general time dependent dynamical systems

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ABSTRACT

In this paper we develop new techniques for revealing geometrical structures in phase space that are valid for aperiodically time dependent dynamical systems, which we refer to as *Lagrangian descriptors*. These quantities are based on the integration, for a finite time, along trajectories of an intrinsic bounded, positive geometrical and/or physical property of the trajectory itself. We discuss a general methodology for constructing Lagrangian descriptors, and we discuss a "heuristic argument" that explains why this method is successful for revealing geometrical structures in the phase space of a dynamical system. We support this argument by explicit calculations on a benchmark problem having a hyperbolic fixed point with stable and unstable manifolds that are known analytically. Several other benchmark examples are considered that allow us the assess the performance of Lagrangian descriptors in revealing invariant tori and regions of shear. Throughout the paper "side-by-side" comparisons of the performance of Lagrangian descriptors with both finite time Lyapunov exponents (FTLEs) and finite time averages of certain components of the vector field ("time averages") are carried out and discussed. In all cases Lagrangian descriptors are shown to be both more accurate and computationally efficient than these methods. We also perform computations for an explicitly three dimensional, aperiodically time-dependent vector field and an aperiodically time dependent vector field defined as a data set. Comparisons with FTLEs and time averages for these examples are also carried out, with similar conclusions as for the benchmark examples.



## 1. Introduction

Since the insights of Poincaré [44] geometrical structures in phase space have played a central role in characterising the global behaviours of dynamical systems. This point of view gives insight into the evolution of qualitatively distinct classes of trajectories without having to explicitly integrate and compare the behaviour of trajectories. This point of view has also been shown to provide much insight into fluid transport and mixing after the recognition that the equations for incompressible fluid particle motion (in the absence of molecular diffusion) are formally equivalent to Hamilton's equations where, in the fluid mechanics context, the stream function plays the role of the Hamiltonian function and the physical space in which the fluid moves plays the role of the phase space of the corresponding Hamilton's equations [2]. This framework for studying fluid transport and mixing is often referred to as the 'dynamical systems approach to Lagrangian

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transport" since the focus is on understanding the "organising structures" in phase space for fluid particle trajectories. References describing this approach for general fluid mechanics are Ottino [43] and Wiggins and Ottino [61], and references specific to geophysical fluid dynamics are Jones and Winkler [22], Wiggins [60], Mancho et al. [33] and Samelson and Wiggins [54].

In this paper we further develop a technique for revealing phase space structure that was proposed and used in Mendoza and Mancho [34], Mendoza et al. [36], Mendoza and Mancho [35] and de la Cámara et al. [11]. This method is based on the computation of arc length of particle trajectories and it is extended by considering the integration of a bounded, positive quantity that is an intrinsic geometrical and/or physical property of the dynamical system along trajectories of the dynamical system, for a finite time. Techniques based on the integration of properties along trajectories have been used in the past. Rypina et al. [53] have proposed methods based on the idea of *complexity of isolated trajectories*. Recently in the computer graphics community, Acar et al. [1] have also used the time-normalised arc length of trajectories for human action recognition. Their work is inspired on the *line integral convolution* discussed by Shi et al. [57] and Cabral and Leedom [8] that advect conserved physical fields along trajectories. The advection of conserved physical fields has been also used in the atmospheric science community since the early 90s. O'Neill et al. [41] and Sutton et al. [58] proposed the *reverse domain filling* method for visualising transport structures in geophysical flows. Recently in the context of realistic oceanic flows the works by Prants et al. [47,48] have highlighted Lagrangian features by computing the time of exit of particles from a given region or the number of changes of the sign of zonal and meridional velocities. Huhn et al. [20] and Prants [46] have addressed similar purposes by measuring the absolute displacements of particles and Prants et al. [49] by counting the number of cyclonic and anticyclonic rotations of particles.

The techniques explained in the current article have been shown to be effective for revealing phase space structures for aperiodically time dependent flows. In Section 2 we describe the general methodology for constructing Lagrangian descriptors, and we emphasise the role of the norm in quantifying the integral of the chosen positive quantity along trajectories. Section 2.1 describes the performance of Lagrangian descriptors in hyperbolic regions, and Section 2.1.1 discusses the benchmark problem of a linear, autonomous vector field in the plane having a hyperbolic saddle point at the origin. This is an excellent example, not only for discussing the issues associate with the performance of Lagrangian descriptors, but also for comparing their performance with finite time Lyapunov exponents (FTLEs) and time averages of particular components of the vector field along trajectories since the stable and unstable manifolds of the hyperbolic point, as well as all trajectories, are known exactly. In this example we are able to illustrate the effect of the integration time and make precise the idea that "singular contours of the Lagrangian descriptors correspond to invariant manifolds", both numerically, and analytically in Section 2.1.2. This example also illustrates why the $L^{\gamma}$ norms, $\gamma > 1$ are not successful in revealing "singular contours" in the same way as the $L^{\gamma}$ norms, $\gamma \leqslant 1$. In Section 2.3 we show that for this example FTLEs completely fail to reveal any phase space. Since the system is linear almost all FTLEs, *for any time*, are equal for every trajectory. A similar failure for finite time averages of a component of the vector field along trajectories is reported in Section 2.3.2. In Section 2.2 we consider another benchmark example – a linear vector field having an elliptic equilibrium point at the origin. As in the example of the linear saddle point, since this system is linear almost all FTLEs are zero, for any time. Hence the FTLE field fails to reveal any phase space structure. The Lagrangian descriptors, on the other hand, reveal the expected phase space structure, consistent with the trajectories. Since this example is linear elliptic point, effects of shear are not considered. In Section 2.2.2 we consider an integrable example in action-angle coordinates (hence all FTLEs are zero, for any time) that not only contain regions of strong shear, but also invariant tori where the "twist condition" breaks down (*i.e.* "twistless tori"). We show that Lagrangian descriptors provide signatures of each of these features. An advantage of the linear saddle point and linear elliptic point examples is that they allow us to "isolate" hyperbolic and elliptic behaviour. In Section 2.4 we consider the forced Duffing equation for three different types of time dependency: (1) no time dependency (forcing), *i.e.* the integrable case, (2) periodic time dependence, and (3) aperiodic time dependence (of a form that we define). In this example hyperbolic and elliptic behaviour are "intermingled" in a complicated manner (that we describe), and this affords us an ideal setting to compare Lagrangian descriptors, FTLEs, and averages of velocity components along trajectories "side-by-side" in different regions of the phase space of the forced Duffing oscillator. In Section 3 we consider a three dimensional vector field from fluid mechanics – the Hill's spherical vortex subjected to a time-aperiodic perturbation-that was studied in Branicki and Wiggins [5]. For this example we show that Lagrangian descriptors provide an efficient way of discovering and visualising detailed geometrical structures in the flow. A comparison is made with FTLEs and averages of components of the vector field along trajectories, and it is shown that these techniques may introduce "artefacts" that obscure the true geometric structures. In Section 4 we consider velocity fields defined as data sets. In particular, we consider a wind driven three-layer quasigeostrophic simulation in a rectangular domain in a "double gyre" configuration exhibiting aperiodic time dependence. This velocity field has been the subject of earlier studies [10,30,33] and therefore allows us to directly compare the performance of Lagrangian descriptors, and FTLEs, with distinguished hyperbolic trajectories (DHTs) and their stable and unstable manifolds. Additionally, we compare the convergence time of Lagrangian descriptors with FTLEs to the stable and unstable subspaces of a given DHT, where we see that the Lagrangian descriptors converge to these structures more quickly. In Section 5 we compare the computational effort required for Lagrangian descriptor computations with FTLE computations, and in Section 6 we summarise our conclusions. In three appendices we provide some additional technical details.