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Pseudospectral reduction of incompressible two-dimensional turbulence

John C. Bowman *, Malcolm Roberts

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

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ABSTRACT

Spectral reduction was originally formulated entirely in the wavenumber domain as a coarse-grained wavenumber convolution in which bins of modes interact with enhanced coupling coefficients. A Liouville theorem leads to inviscid equipartition solutions when each bin contains the same number of modes. A pseudospectral implementation of spectral reduction which enjoys the efficiency of the fast Fourier transform is described. The model compares well with full pseudospectral simulations of the two-dimensional forced-dissipative energy and enstrophy cascades.

1. Introduction

Turbulence researchers have long sought to reduce the number of degrees of freedom that must be retained in simulations of turbulent phenomena [1–10]. About 10 years ago a novel decimation method called spectral reduction [11], which accounts for the deleted modes by coarse-graining the nonlinearity, was developed in collaboration with Prof. Philip Morrison, to whom we dedicate this paper, on the occasion of his 60th birthday.

Spectral reduction is formulated as a coarse-grained wavenumber convolution over bins of modes that interact with enhanced coupling coefficients. The approximation possesses many desirable properties: coarse-grained analogues of primitive conservation laws, a Liouville theorem, plausible moment balance equations that resemble the exact moment balances, and a control parameter (bin size) that can be varied to increase accuracy incrementally, until the exact dynamics is recovered. In the case where each bin contains an identical number of modes, spectral reduction recovers (for mixing dynamics) the well known inviscid equipartition solutions of spectrally truncated turbulence [12–16]. In particular, in two dimensions, one obtains an equipartition of a linear combination of the modal energies and enstrophies [17], as shown in Fig. 1.

Two serious limitations have prevented spectral reduction from being adopted for general use. First, if the bins contain different numbers of modes (a case of practical significance, e.g. for subgrid modelling), the Liouville theorem, together with the coarse-grained conservation laws, incorrectly leads to equipartitions of bin, rather than modal, energies and enstrophies [16]. Second, because spectral reduction coarse grains the convolution arising from the advective nonlinearity, it was originally formulated entirely in the wavenumber domain. This limited its use to situations where the wavenumber reduction factor is large enough to offset the inefficiency of spectral methods relative to their pseudospectral counterparts.

In this work, we report on a recent pseudospectral implementation of spectral reduction that exploits the efficiency of the fast Fourier transform (FFT). The resulting implementation of *pseudospectral reduction* agrees well with full pseudospectral simulations of the energy and enstrophy cascades of two-dimensional (2D) forced-dissipative turbulence.

* Corresponding author.
E-mail address: bowman@math.ualberta.ca (J.C. Bowman).
URL: http://www.math.ualberta.ca/~bowman (J.C. Bowman).

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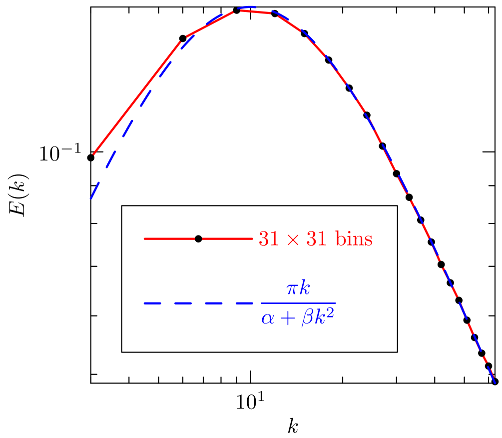

Fig. 1. Inviscid equipartition of a 31×31 pseudospectrally reduced simulation with radix $r = 3$ to the theoretically expected curve $\pi k/(\alpha + \beta k^2)$, where the constants α and β are Lagrange multipliers.

Let us begin by writing in Fourier space the 2D vorticity equation

$$\frac{\partial \omega_{\boldsymbol{k}}}{\partial t} + \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} = \sum_{\boldsymbol{p},\boldsymbol{q}} \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* + f_{\boldsymbol{k}} \xi(t), \tag{1}$$

where $\nu_{\boldsymbol{k}} \doteq \nu k^2$ models molecular viscosity, $\epsilon_{\boldsymbol{kpq}} \doteq (\hat{\boldsymbol{z}} \cdot \boldsymbol{p} \times \boldsymbol{q}) \delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0}$ is antisymmetric under permutation of any two indices, $f_{\boldsymbol{k}}$ is an external stirring amplitude, and $\xi(t)$ represents a unit Gaussian stochastic white-noise process ($\doteq$ is used to emphasize a definition). According to a result by Novikov, the steady-state enstrophy injection rate is given by $\epsilon_Z = \frac{1}{2}\sum_{\boldsymbol{k}} |f_{\boldsymbol{k}}|^2$ [18].

We introduce a uniform coarse-grained grid that partitions the two-dimensional wavenumber domain into bins, each containing the same number r^2 of Fourier wave vectors (assuming for simplicity that the binning reductions r in each direction are identical). The bins are labeled by capital letters to distinguish them from the underlying Fourier wave vectors, which we represent by lower-case letters. To each bin $\boldsymbol{K}$ on this grid, we associate the coarse-grained vorticity $\Omega_{\boldsymbol{K}} \doteq \frac{1}{r^2} \sum_{\boldsymbol{k} \in \boldsymbol{K}} \omega_{\boldsymbol{k}}$ and assign a characteristic Fourier wave vector $\boldsymbol{K}$ (whether $\boldsymbol{K}$ refers to a bin or its characteristic wave vector should always be clear from context).

On introducing the bin averaging operations

$$\langle f_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \doteq \frac{1}{r^2} \sum_{\boldsymbol{k} \in \boldsymbol{K}} f_{\boldsymbol{k}} \quad \text{and} \quad \langle f_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \doteq \frac{1}{r^6} \sum_{\boldsymbol{k} \in \boldsymbol{K}} \sum_{\boldsymbol{p} \in \boldsymbol{P}} \sum_{\boldsymbol{q} \in \boldsymbol{Q}} f_{\boldsymbol{kpq}}, \tag{2}$$

we may write the exact evolution equation for $\Omega_{\boldsymbol{K}}$ as

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \omega_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} = r^4 \sum_{\boldsymbol{P},\boldsymbol{Q}} \left\langle \frac{\epsilon_{\boldsymbol{kpq}}}{q^2} \omega_{\boldsymbol{p}}^* \omega_{\boldsymbol{q}}^* \right\rangle_{\boldsymbol{KPQ}} + \langle f_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \xi(t). \tag{3}$$

Spectral reduction [11] approximates the evolution of $\Omega_{\boldsymbol{K}}$ solely in terms of coarse-grained variables:

$$\frac{\partial \Omega_{\boldsymbol{K}}}{\partial t} + \langle \nu_{\boldsymbol{k}} \rangle_{\boldsymbol{K}} \Omega_{\boldsymbol{K}} = r^4 \sum_{\boldsymbol{P},\boldsymbol{Q}} \frac{1}{Q^2} \langle \epsilon_{\boldsymbol{kpq}} \rangle_{\boldsymbol{KPQ}} \Omega_{\boldsymbol{P}}^* \Omega_{\boldsymbol{Q}}^* + F_{\boldsymbol{K}} \xi(t), \tag{4}$$

where $F_{\boldsymbol{K}}$ represents a coarse-grained stirring force. In this work we choose $F_{\boldsymbol{K}} = 2\epsilon_Z f_K / \sqrt{\sum_{\boldsymbol{K}} |f_K|^2}$ to inject exactly ϵ_Z units of enstrophy in a steady state. In the absence of forcing and dissipation, Eq. (4) is readily seen to conserve the coarse-grained energy $\frac{1}{2}\sum_{\boldsymbol{K}} K^{-2} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}$ and enstrophy $\frac{1}{2}\sum_{\boldsymbol{K}} |\Omega_{\boldsymbol{K}}|^2 \Delta_{\boldsymbol{K}}$. However, since the $\delta_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q},0}$ factor appearing in the nonlinear coefficient $\epsilon_{\boldsymbol{kpq}}$ is averaged in Eq. (4) over $\boldsymbol{k}$, $\boldsymbol{p}$, and $\boldsymbol{q}$, the resulting equation is no longer a convolution. This means that Eq. (4) cannot be solved directly by the usual pseudospectral method. Nevertheless, in the case of uniform binning a generalization of the pseudospectral method can be developed to efficiently implement Eq. (4). To illustrate this, we first develop in the following section an efficient method for computing coarse-grained convolutions.

2. One-dimensional coarse-grained convolution

On defining the Nth primitive root of unity, $\zeta_N \doteq \exp(2\pi i/N)$, the one-dimensional *backwards discrete Fourier transform* of a complex vector $\{F_k : k = 0, \ldots, N-1\}$ may be written as