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Nontwist symplectic maps in tokamaks

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ABSTRACT

We review symplectic nontwist maps that we have introduced to describe Lagrangian transport properties in magnetically confined plasmas in tokamaks. These nontwist maps are suitable to describe the formation and destruction of transport barriers in the shearless region (i.e., near the curve where the twist condition does not hold). The maps can be used to investigate two kinds of problems in plasmas with non-monotonic field profiles: the first is the chaotic magnetic field line transport in plasmas with external resonant perturbations. The second problem is the chaotic particle drift motion caused by electrostatic drift waves. The presented analytical maps, derived from plasma models with equilibrium field profiles and control parameters that are commonly measured in plasma discharges, can be used to investigate long-term transport properties.

1. Introduction

Symplectic maps have been for a long time used as models of Hamiltonian systems [1]. Such maps may arise from intersections of phase space trajectories with a given surface of section, or also may appear as stroboscopic samplings of a trajectory of a time-dependent system at fixed time intervals [2]. If the time dependence is a periodic sequence of delta function kicks it is possible to obtain explicit formulas for the maps [1,3]. Symplectic maps are convenient systems to investigate, inasmuch we can compute a large number of iterations using a short CPU time and practically without propagating numerical errors. This is particularly important in investigations of Hamiltonian transport requiring computation of phase space trajectories during a very long time interval.

Traditionally studies of symplectic maps deal with twist maps for which the rotation number varies monotonically. Many important theoretical results, like Kolmogorov–Arnold–Moser (KAM) and Poincaré–Birkhoff theorems, and Aubry–Mather theory are stated for two-dimensional symplectic maps [2].

Recently there is a growing interest in nontwist maps, for which the twist condition is not fulfilled for all points in the domain of interest [2,4]. Nontwist maps are of mathematical interest because only a few rigorous results exist for them [5–9].

Moreover, nontwist maps also appear in many dynamical systems of physical interest, often related to continuous systems like fluids and plasmas, such as magnetic field lines in toroidal plasma devices with reversed magnetic shear [4,10–14], advection by incompressible shear flows [15], traveling waves in geophysical zonal flows [16,17], and the $\mathbf{E} \times \mathbf{B}$ drift motion of charged particles in a magnetized plasma under the action of a time-periodic electric field from an

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electrostatic wave [18–21]. Nontwist maps have also been investigated in particle motions described by certain Hamiltonians in molecular physics [22,23].

Here, we review examples of nontwist maps that we have proposed, some of them in collaboration with Phil Morrison. For magnetically confined plasmas in tokamaks, nontwist maps are introduced to describe the presence of Lagrangian chaos at the plasma edge. Initially we consider the Lagrangian description of magnetic field lines in tokamaks [10,21]. In this description the field lines for a confined plasma with a non-monotonic electric current density are mapped into a two-dimensional Poincaré surface of section with a nontwist rotation number radial profile. This nontwist condition at the plasma edge creates a barrier to the field line that attenuates the plasma transport, so improving plasma confinement [24].

In the second place we consider the chaotic motion of particles at the plasma edge in an equilibrium electric field with non-monotonic radial profile perturbed by drift waves [16,18,20]. For these profiles the particle trajectories can be also mapped in to a two-dimensional Poincaré section with a nontwist rotation number radial profile, a result similar to that obtained for magnetic field lines. To illustrate that we present a new nontwist drift map obtained for a nonmonotonic electric potential that creates a transport barrier in the nontwist region.

This paper is organized as follows: in Section 2 we introduce the symplectic nontwist maps through its most studied example, the standard nontwist map introduced by Phil Morrison and his collaborators. Sections 3 and 4 present, respectively, the nontwist maps we derived for magnetic field lines and particle transport in tokamaks. Our conclusions are left to the final Section.

2. Nontwist symplectic maps

Let (x_n, y_n) be the normalized angle and action variables, respectively, of a phase space trajectory at its nth piercing with a Poincaré surface of section, such that $y_n \in \mathbb{R}$ and $x_n \in (-1/2, 1/2]$. We will consider two-dimensional symplectic maps of the form

$$y_{n+1} = y_n + f(x_n), \tag{1}$$
$$x_{n+1} = x_n - g(y_{n+1}), \quad (mod 1) \tag{2}$$

where $f(x_n)$ is a period-1 function of its argument, standing for the perturbation strength. We assume that the time-dependence of the perturbing term can be modeled by a periodic delta function. Moreover $g(y_{n+1})$ is the winding number of the unperturbed trajectories, its derivative being the so-called shear function. If g is a monotonically increasing or decreasing function the corresponding shear does not change sign, giving the twist condition:

$$|g'(y_{n+1})| = \left|\frac{\partial x_{n+1}}{\partial y_n}\right| \geqslant c > 0, \tag{3}$$

where $c \in \mathbb{R}$.

However, for a non-monotonic g-profile the twist condition is not satisfied at some point y_s and $g'(y_s) = 0$ determines the shearless curve. If the winding number $g(y)$ has only one extremum at the shearless curve $y = y_S$, we can expand this function in the vicinity of this point. The lowest-order approximation for $g(y)$ is thus a quadratic function. On keeping one Fourier mode in the perturbation term of (1) and (2), we obtain the standard non-twist map (SNTM) [16]

$$x_{n+1} = x_n + a(1 - y_{n+1}^2), \tag{4}$$
$$y_{n+1} = y_n - b\sin(2\pi x_n), \tag{5}$$

where $a \in (0, 1)$, and $b > 0$.

The mathematical properties of SNTM have been extensively investigated by Phil Morrison and his collaborators over the past two decades [25]. In the following we outline some of these properties, referring to the literature for a more complete coverage of them. In the unperturbed case ($b = 0$) the twist condition (3) is violated at the point $y_S = 0$, what defines a shearless curve $\{y = y_S = 0\}$. The quadratic form of g around $y_S = 0$ leads to two invariant curves located at $y = \pm y_0$ and with the same winding number $a(1 - y_0^2)$ at both sides of the shearless curve.

As the perturbation becomes nonzero ($b \neq 0$) two periodic island chains appear at the two invariant curve locations, and the former shearless curve becomes a shearless invariant tori separating these two island chains [16]. There are also chaotic layers attached to the "separatrices" of both island chains, as expected from the presence of homoclinic crossings therein. These chaotic layers are not connected, though, as far as there are invariant curves near the shearless invariant tori acting as dikes, preventing global transport [26].

A representative example of this scenario is provided by Fig. 1(a), where a phase portrait of the SNTM is depicted for $b = 0.410$ and $a = 0.700$. We observe two island chains with three islands each and the same winding number. The local maxima of the perturbed winding number profile define a shearless invariant curve, which existence can be inferred between the two island chains. The island chains bordering the shearless invariant curve are transport barriers, since chaotic trajectories above and below do not mix. If, however, the parameters are further modified, another noteworthy feature of nontwist maps can emerge, depending on the parameter space region. In one scenario (generic reconnection) the island chains with the same winding number approach each other and their unstable and stable invariant manifolds suffer a (non-dissipative)