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The destruction of tori in volume-preserving maps

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ABSTRACT

Invariant tori are prominent features of symplectic and volume-preserving maps. From the point of view of chaotic transport the most relevant tori are those that are barriers, and thus have codimension one. For an n-dimensional volume-preserving map, such tori are prevalent when the map is nearly "integrable," in the sense of having one action and $n-1$ angle variables. As the map is perturbed, numerical studies show that the originally connected image of the frequency map acquires gaps due to resonances and domains of nonconvergence due to chaos. We present examples of a three-dimensional, generalized standard map for which there is a critical perturbation size, ε_c, above which there are no tori. Numerical investigations to find the "last invariant torus" reveal some similarities to the behavior found by Greene near a critical invariant circle for area preserving maps: the crossing time through the newly destroyed torus appears to have a power law singularity at ε_c, and the local phase space near the critical torus contains many high-order resonances.

1. Introduction

Volume-preserving maps are appropriate models for many systems including fluid flows [1–7], granular mixers [8], magnetic field line flows [9–11], and even the motion of comets perturbed by a planet on an elliptical orbit [12]. Volume-preserving dynamics has some similarities to symplectic dynamics; however, though every symplectic map is volume preserving, the converse is only true in two-dimensions.

This paper is concerned with the effects of perturbation and resonance on invariant tori; such tori are especially common in the integrable case. In the context of Hamiltonian systems and symplectic maps, integrability is synonymous with Liouville's definition: a d-degree of freedom system is integrable when it has a set of d, almost-everywhere independent, involutory invariants. The involution property of the invariants implies that they also generate an Abelian group of symmetries that preserve the invariants, and therefore that the (compact) integral manifolds are tori. An integrable $2d$-dimensional symplectic map can be written (at least locally) in terms of angle-action variables $(\theta, J) \in \mathbb{T}^d \times \mathbb{R}^d$ as

$$\begin{aligned} \theta' &= \theta + \Omega(J), \\ J' &= J, \end{aligned} \tag{1}$$

where we will take $\mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$ [13]. The concept of integrability is perhaps less well-formulated for the volume-preserving case. However, it seems quite natural to use Bogoyavlenskij's concept of *broad integrability* [14,15]. Roughly speaking, a system of n ODEs is broadly integrable if it has k independent invariants and d commuting symmetries that preserve the invariants, where $k + d = n$. We propose a similar definition for a map—the integrable case corresponds to the form (1) as well, but now $(\theta, J) \in \mathbb{T}^d \times \mathbb{R}^k$ [16].

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It is natural to study (1) on the universal cover, letting $(x,z) \in \mathbb{R}^d \times \mathbb{R}^k$, so that $(\theta, J) = (x \bmod 1, z)$. We study a simple class of perturbations of (1), which for the lift becomes

$$\begin{aligned} x' &= x + \Omega(z'), \\ z' &= z - \varepsilon g(x), \end{aligned} \tag{2}$$

where the "force," g, is periodic, i.e., $g(x+m) = g(x)$ for any $m \in \mathbb{Z}^d$. Since Ω is evaluated at z' in (2), this map is a volume-preserving diffeomorphism for any smooth functions Ω and g.[1] It is exact volume preserving when g has zero average—or equivalently the zero Fourier component of g vanishes [17]. Since this is the only case for which (2) can have rotational tori—that is, tori homotopic to the tori of (1)—that are invariant, we will make this assumption. The unperturbed rotation vector (or frequency) map:

$$\Omega : \mathbb{R}^k \to \mathbb{R}^d \tag{3}$$

plays an especially important role in the dynamics of (1). For the integrable map, the forward orbit $\{(x_t, z_t) : t \in \mathbb{N}\}$ of each initial condition (x_0, z_0) has a rotation vector

$$\omega(x_0, z_0) = \lim_{T\to\infty} \frac{x_T - x_0}{T} \tag{4}$$

given by the unperturbed map $\Omega(z_0)$. If $\Omega(z_0)$ is incommensurate, see Section 2, the orbit densely covers a d-dimensional torus; by contrast, when the rotation vector is resonant an orbit densely covers one or more lower dimensional tori.

When the map (1) is perturbed, many of its d-tori are immediately destroyed; however, KAM theory implies that there will still be a large set of invariant tori if the perturbation is small enough and smooth enough and the frequency satisfies a nondegeneracy condition. This is rigorously true for $k = d$ when the perturbed map is exact symplectic and satisfies a Hölder condition (i.e. is C^{3+h} for some h > 0), and Ω satisfies a nondegeneracy condition such as the *twist condition*

$$\det D\Omega \geqslant c > 0, \tag{5}$$

see, e.g., [18,19].

Of course, when $k < d$, the number of actions is smaller than the number of angles and the matrix $D\Omega$ is no longer square. The "nicest" case corresponds to rank $(D\Omega) = k$ implying that the image of Ω is an immersed k-dimensional submanifold.

Maps of the form (1) with $k = 1$, so-called *one-action* maps [20], have codimension-one invariant tori. KAM theory implies that codimension-one tori are robust features of nearly-integrable, analytic one-action maps [21,22]. These theorems assume that $\Omega \in C^{d+1}$ and satisfies a nondegeneracy condition of the form

$$\det(D\Omega, D^2\Omega, \ldots D^d\Omega) \geqslant c > 0, \tag{6}$$

similar to that used by Rüssmann [23,24].

By contrast, when $1 < k < d$ the invariant $d-k$ dimensional tori of (1) need not be as robust. For example when there are two actions and one angle, almost all of the one-dimensional tori can apparently be immediately destroyed even under a smooth arbitrarily small perturbation [25].

Though codimension-one tori are commonly observed in near-integrable, one-action maps, they are often destroyed by resonant bifurcations as the perturbation grows. The nature of these bifurcations is strongly influenced by the form of the frequency map. Even when the map satisfies (6) perturbations of (1) can have many of the features of symplectic maps that do not satisfy the twist condition [26], see Section 3. In Section 4, we will investigate which of these persistent tori is most robust.

2. Frequency maps and resonance

A rotation vector $\omega \in \mathbb{R}^d$ is "resonant" when there exists an $(m,n) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{0,0\}$ such that

$$m \cdot \omega = n. \tag{7}$$

The *resonance module* of a given ω is a sub-lattice of $\mathbb{Z}^d$ defined by

$$\mathcal{L}(\omega) \equiv \{m \in \mathbb{Z}^d : m \cdot \omega \in \mathbb{Z}\}. \tag{8}$$

The dimension of this module (the number of independent m-vectors in $\mathcal{L}$) is the *rank* of the resonance. An incommensurate or nonresonant rotation vector corresponds to the rank-zero case, $\mathcal{L} = \{0\}$. The *order* of a resonance is the length of the smallest nonzero vector in $\mathcal{L}$, though of course this depends upon the norm used; we will typically use the sup-norm. The set of resonant frequencies

$$\mathcal{R} \equiv \left\{\omega \in \mathbb{R}^d : m \cdot \omega = n \text{ for some } (m,n) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{0,0\}\right\}$$

is the resonance web; it is a dense subset of $\mathbb{R}^d$.

[1] When $d = k$, (2) is symplectic with the form $dx \wedge dz$ only if $D\Omega$ and Dg are symmetric matrices.