Commun Nonlinear Sci Numer Simulat 44 (2017) 11–18

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Bäcklund transformation classification, integrability and exact solutions to the generalized Burgers'–KdV equation[☆]

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ARTICLE INFO

Article history:
Received 29 January 2016
Accepted 22 July 2016
Available online 29 July 2016

MSC:
37K10
37K35
35C05,

Keywords:
Direct method
Homogeneous balance principle
Bäcklund transformation
Integrability
Exact solution

ABSTRACT

This paper is concerned with the Bäcklund transformations (BTs) of the nonlinear evolution equations (NLEEs). Based on the homogeneous balance principle (HBP), the existence of the BT of the generalized Burgers'–KdV (B–KdV) equation is classified, then the BTs of the nonlinear equations are given. In general, the method can be used to construct BTs of the nonlinear evolution equations in polynomial form. Furthermore, the integrability and exact explicit solutions to the nonlinear equations are investigated.

1. Introduction

As is well known, the Bäcklund transformation (BT) is an intrinsic property shared by integrable system. Once a Bäcklund transformation of a nonlinear system is obtained, the complete integrability of the system is proved. Furthermore, based on the Bäcklund transformation, many other integrable properties of the system can be considered, such as the Hamiltonian structure, conservation laws (CLs) and soliton solutions. Hence, the Bäcklund transformations of nonlinear systems have long been and will continue to be one of the dominant themes in both nonlinear theory and mathematical physics due to its fundamental importance. In the past few decades, there are noticeable progress in this field, and various methods have been developed, such as the inverse scattering transformation (IST), Darboux transformation (DT) and Lax pair (LP) methods [1–8], the Lie symmetry analysis [9–12], Painlevé test [12–16], and so on.

In this paper, we investigate the Bäcklund transformations of the generalized Burgers'–KdV (B–KdV) equation as follows:

$$u_t + \alpha u^p u_x + \beta u_{xx} + \gamma u_{xxx} = 0, \tag{1.1}$$

☆ This work is supported by the National Natural Science Foundation of China under grant Nos. 11171041 and 11505090, Research Award Foundation for Outstanding Young Scientists of Shandong Province under grant No. BS2015SF009, and the high-level foundation of Liaocheng University under grant No. 31805.

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http://dx.doi.org/10.1016/j.cnsns.2016.07.022
1007-5704/

where $u = u(x, t)$ denotes the unknown function with respect to the space variable x and time t, the parameters $\alpha \neq 0$, β and γ are all arbitrary constants, which denote the nonlinear, dissipative and dispersive coefficients, respectively, p is a positive integer.

We note that Eq. (1.1) is a nonlinear evolution equation (NLEE), which includes a lot of famous nonlinear evolution equations as its special cases. For example, if $p = 1$, then Eq. (1.1) is the classical B–KdV equation as follows

$$u_t + \alpha u u_x + \beta u_{xx} + \gamma u_{xxx} = 0. \tag{1.2}$$

In particular, if $\beta = 0$, then this equation is the Korteweg–de Vries (KdV) equation

$$u_t + \alpha u u_x + \gamma u_{xxx} = 0. \tag{1.3}$$

If $p = 2$ and $\beta = 0$, then Eq. (1.1) is the modified Korteweg–de Vries (mKdV) equation

$$u_t + \alpha u^2 u_x + \gamma u_{xxx} = 0. \tag{1.4}$$

If $p = 1$ and $\gamma = 0$, then Eq. (1.1) is reduced to the Burgers' equation (BE)

$$u_t + \alpha u u_x + \beta u_{xx} = 0. \tag{1.5}$$

If $p = 2$ and $\gamma = 0$, then Eq. (1.1) is reduced to the modified Burgers' equation (mBE) [17]

$$u_t + \alpha u^2 u_x + \beta u_{xx} = 0, \tag{1.6}$$

and so on.

More generally, we shall consider the evolution equations in polynomial form as follows

$$u_t = P[u], \tag{1.7}$$

where $P[u] = P(x, t, u, u_x, \ldots)$ is a polynomial with respect to their variables.

The above nonlinear evolution equations are of great importance in nonlinear wave theory, integrable system and physical applications [1–7,17,18]. In [11,12], we studied the Bäcklund transformations and exact solutions to some NLEEs by the symmetry analysis method. Recently, the Bäcklund transformations and exact solutions to the other nonlinear equations are considered based on the Painlevé test [15,16]. It is known that the homogeneous balance principle (HBP) is a general and systematic method for dealing with exact explicit solutions to the nonlinear partial differential equations (NLPDEs) in polynomial forms [19,20], and the Bäcklund transformations of such nonlinear equations can be constructed also. However, for some other nonlinear equations such as the B–KdV and modified Burgers' Eqs. (1.2) and (1.6), the homogeneous balance principle is of no help. The main purpose of this paper is to classify the generalized B–KdV Eq. (1.1) for the first time, and give the Bäcklund transformations, exact solutions and integrability. More generally, the method can be used to construct the Bäcklund transformations of the generalized evolution Eq. (1.7).

The rest of this paper is organized as follows: In Section 2, based on the homogeneous balance principle, the existence of the Bäcklund transformations of the nonlinear Eq. (1.1) is classified, then the Bäcklund transformations of the nonlinear evolution equations are constructed. Especially, by construction a Bäcklund transformations of an auxiliary equation, the so called generalized Bäcklund transformation of the the modified Burgers' equation is provided for the first time. In Section 3, the integrability and exact explicit solutions to the nonlinear equations are investigated by using the Bäcklund transformations. In addition, a shock wave solution to the B–KdV equation is obtained by the homogeneous balance principle. Finally, we summarize our new finding and give some remarks in Section 4.

2. BT classification of the generalized B–KdV equation

In general, for a nonlinear system, the Bäcklund transformations can be derived from different procedures, such as the Painlevé test, Lax pairs, Darboux transformations, Hirota's bilinear method and symmetry analysis method, etc. In this section, we shall introduce a direct analytic method for constructing Bäcklund transformations of the above nonlinear equations, the idea of our method is based on the homogenous balance principle [18,19]. Generally, we suppose that Eq. (1.1) has a solution in the following form:

$$u = \frac{\partial^{m+n} f(h)}{\partial x^m \partial t^n} + v, \tag{2.1}$$

where m and n are nonnegative integers, $f = f(h)$ is a composite function with respect to f and $h = h(x, t)$, which to be determined later, $v = v(x, t)$ is a given solution to Eq. (1.1). If we determine the functions $f = f(h)$ and $h = h(x, t)$ (under some condition sometimes), then the Bäcklund transformation of the equation is obtained.

First of all, in view of the arbitrary constant γ in Eq. (1.1), we have the following cases:

(a) $\gamma \neq 0$; or
(b) (b) $\gamma = 0$.

In what follows, we discuss the above cases one by one.