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Short communication

No violation of the Leibniz rule. No fractional derivative

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ABSTRACT

We demonstrate that a violation of the Leibniz rule is a characteristic property of derivatives of non-integer orders. We prove that all fractional derivatives $\mathcal{D}^\alpha$, which satisfy the Leibniz rule $\mathcal{D}^\alpha(fg) = (\mathcal{D}^\alpha f)g + f(\mathcal{D}^\alpha g)$, should have the integer order $\alpha = 1$, i.e. fractional derivatives of non-integer orders cannot satisfy the Leibniz rule.

1. Introduction

Fractional derivatives of non-integer orders [1,2] have wide applications in physics and mechanics [4–13]. The tools of fractional derivatives and integrals allow us to investigate the behavior of objects and systems that are characterized by power-law non-locality, power-law long-term memory or fractal properties.

There are different definitions of fractional derivatives such as Riemann–Liouville, Riesz, Caputo, Grünwald-Letnikov, Marchaud, Weyl, Sonin-Letnikov and others [1,2]. Unfortunately all these fractional derivatives have a lot of unusual properties. The well-known Leibniz rule $\mathcal{D}^\alpha(fg) = (\mathcal{D}^\alpha f)g + f(\mathcal{D}^\alpha g)$ is not satisfied for differentiation of non-integer orders [1]. For example, we have the infinite series

$$\mathcal{D}^\alpha(fg) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} (\mathcal{D}^{\alpha-k} f)(D^k g) \tag{1}$$

for analytic functions on $[a, b]$ (see Theorem 15.1 in [1]), where $\mathcal{D}^\alpha$ is the Riemann–Liouville derivative, D^k is derivative of integer order k. Note that the sum is infinite and contains integrals of fractional order for $k > [\alpha] + 1$. Formula (1) first appeared in the paper by Liouville [3] in 1832.

The unusual properties lead to some difficulties in application of fractional derivatives in physics and mechanics. There are some attempts to define new type of fractional derivative such that the Leibniz rule holds (for example, see [15–17]).

In this paper we proof that a violation of the Leibniz rule is one of the main characteristic properties of fractional derivatives. We state that linear operator $\mathcal{D}^\alpha$ that can be defined on $C^2(U)$, where $U \subset \mathbb{R}^1$, such that it satisfied the Leibniz rule cannot have a non-integer order α. In other words, a fractional derivative that satisfies the Leibniz rule is not fractional. It should have integer order.

2. Hadamard's theorem

We denote by $C^m(U)$ a space of functions $f(x)$, which are m times continuously differentiable on $U \subset \mathbb{R}^1$. Let $D_x^1 = d/dx: \ C^m(U) \to C^{m-1}(U)$ be a usual derivative of first order with respect to coordinate x.

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It is well-known the following Hadamard's theorem [14].

Hadamard's Theorem. *Any function $f(x) \in C^1(U)$ in a neighborhood U of a point x_0 can be represented in the form*

$$f(x) = f(x_0) + (x - x_0)g(x), \tag{2}$$

where $g(x) \in C^m(U)$.

Proof. Let us consider the function

$$F(t) = f(x_0 + (x - x_0)t). \tag{3}$$

Then $F(0) = f(x_0)$ and $F(1) = f(x)$. The Newton–Leibniz formula gives

$$F(1) - F(0) = \int_0^1 dt\,(D_t^1 F)(t) = \int_0^1 dt\,(D_x^1 f)(x_0 + (x - x_0)t)\,(x - x_0) = (x - x_0)\int_0^1 dt\,(D_x^1 f)(x_0 + (x - x_0)t). \tag{4}$$

We define the function

$$g(x) = \int_0^1 dt\,(D_x^1 f)(x_0 + (x - x_0)t). \tag{5}$$

As the result, we have proved representation (2). □

3. Algebraic approach to fractional derivatives

We consider fractional derivatives $\mathcal{D}^\alpha$ of non-integer orders α by using an algebraic approach. Special forms of fractional derivatives are not important for our consideration. We take into account the property of linearity and the Leibniz rule only.

For the operator $\mathcal{D}^\alpha$ we will consider the following conditions.

(1) $\mathbb{R}$-linearity:

$$\mathcal{D}_x^\alpha(c_1 f(x) + c_2 g(x)) = c_1(\mathcal{D}_x^\alpha f(x)) + c_2\,(\mathcal{D}_x^\alpha g(x)), \tag{6}$$

where c_1 and c_2 are real numbers. Note that all known fractional derivatives are linear [1,2].

(2) The Leibniz rule:

$$\mathcal{D}_x^\alpha(f(x)\,g(x)) = (\mathcal{D}_x^\alpha f(x))\,g(x) + f(x)\,(\mathcal{D}_x^\alpha g(x)). \tag{7}$$

(3) If the linear operator satisfies the Leibniz rule, then the action on the unit (and on a constant function) is equal to zero:

$$\mathcal{D}_x^\alpha 1 = 0. \tag{8}$$

Let us proof the following theorem.

Theorem ("No violation of the Leibniz rule. No fractional derivative"). *If an operator $\mathcal{D}_x^\alpha$ can be applied to functions from $C^2(U)$, where $U \subset \mathbb{R}^1$ be a neighborhood of the point x_0, and conditions (6) and (7) are satisfied, then the operator $\mathcal{D}_x^\alpha$ is the derivative D_x^1 of integer (first) order, i.e. it can be represented in the form*

$$\mathcal{D}_x^\alpha = a(x)\,D_x^1, \tag{9}$$

where $a(x)$ are functions on $\mathbb{R}^1$.

Proof

(1) Using Hadamard's theorem for the function $g(x)$ in the decomposition (2), the function $f(x)$ for $x \in U$ can be represented in the form

$$f(x) = f(x_0) + (x - x_0)g(x_0) + (x - x_0)^2\,g_2(x), \tag{10}$$

where $g_2(x) \in C^2(U)$, and $U \subset \mathbb{R}^1$ is a neighborhood of the point x_0.
Applying to equality (10) the operator D_x^1 and use $D_x^1 f(x_0) = 0$, we get

$$(D_x^1 f)(x) = g(x_0) + 2(x - x_0)\,g_2(x) + (x - x_0)^2\,(D_x^1 g_2)(x). \tag{11}$$

Then

$$(D_x^1 f)(x_0) = g(x_0).$$