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Numerical solutions of the *m*-membranes problem ☆

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ABSTRACT

In this paper numerical approximation for the *m*-membrane problem is considered. We make a change of variables that leads to a different expression of the quadratic functional that allows after discretizing the problem to reformulate it as finite dimensional bound constrained quadratic problem. To our knowledge this is the first paper on numerical approximation of the *m*-membrane problem. We reformulate the *m*-membrane problem as a bound constraint quadratic minimization problem. The bound constraint quadratic form is solved with the gradient projection method.

1. Introduction

1.1. Mathematical formulation

Assume that Ω is a bounded open subset of $\mathbb{R}^n$ with smooth boundary and Δ denotes the Laplace operator. Let $f_1,\dots,f_m$ be m distributions in $H^{-1}(\Omega)$ (the dual of the usual Sobolev space $H_0^1(\Omega)$) and let $g_1,\dots,g_m$ be m functions in $H^1(\Omega)$ that in the trace sense satisfy

$$g_1 \geqslant g_2 \geqslant \cdots \geqslant g_m \quad \text{on } \partial\Omega.$$

Set

$$K = \{(u_1,\dots,u_m) : u_i - g_i \in H_0^1(\Omega), \quad u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m, \quad \text{in } \Omega\}.$$

In what follows we will denote by $\langle , \rangle$ the duality between H^{-1} and H_0^1. In particular, if $u \in H^1(\Omega)$ and $v \in H_0^1(\Omega)$ then

$$\langle -\Delta u, v\rangle = \int_\Omega \frac{\partial u}{\partial x_i}\cdot\frac{\partial v}{\partial x_i}dx.$$

In this setting the *m*-membranes problem is to find a solution $u = (u_1,\dots,u_m) \in K$ of the variational inequality

$$\sum_{k=1}^m \langle -\Delta u_k, u_k - v_k\rangle \geqslant \sum_{k=1}^m \langle f_k, u_k - v_k\rangle \quad \forall v = (v_1,\dots,v_m) \in K. \tag{1}$$

Lemma 1.1 [6]. *Under the above assumptions there exists a unique solution $u = (u_1,\dots,u_m)$ to the variational inequality (1).*

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Remark 1. Assume $m = 1$. Then the problem can be formulated as follows: Let V be a real Hilbert space with bilinear form $a(\cdot,\cdot): V \times V \to \mathbb{R}, K$ be a closed convex nonempty subset of V, and $L: V \to \mathbb{R}$ a continuous linear functional. Then we have the following elliptic variational inequality:

Find $u \in K \subset V$ such that u is a solution of the problem

$$a(u, v-u) \geqslant L(v-u) \quad \forall v \in K.$$

The existence and uniqueness of the solution of the m-membranes problem has been studied in [5,6]. The case $m = 2$ has been studied in [3] for the uniformly elliptic linear case. For the case of $m > 2$, Chipot and Caffarelli studied the regularity in [6]. In [17] Silvestre studied regularity of the free boundary in the two membranes problem. He proved that around any point the free boundary is either a $C^{1,\alpha}$ surface or a cusp, as in the obstacle problem. In addition $C^{1,1}$ regularity for the pair of functions solving the problem is shown. In [2], Azevedo, Rodrigues and Santos studied the regularity of the solution of the variational inequality for the problem of m-membranes in equilibrium with a degenerate operator of p-Laplacian type, $1 < p < \infty$. Finally, Lindgren and Razani in [14] proved the optimal growth of the consecutive differences $u_i - u_{i+1}$ and that the free boundaries $\partial\{u_i > u_{i+1}\}$ have zero Lebesgue measure, under some assumptions on the functions f_i. To see more about background of physical problems and related contacting membranes one can check [2,3,6].

The main contribution of this paper is to obtain numerical solution of the m-membranes problem. For this we rewrite the problem as bound constraint minimization problem. Our motivation is based on the fact that if the number of membranes is $m = 2$ then the difference $u_1 - u_2$ will be a one-phase obstacle problem while the summation gives a partial differential equation. For any given m we reformulate the minimization as quadratic form such that the constraints $u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m$ will be nonnegative in the new variables. To make this work complete and self contained some well known methods like working set methods, Polyak's method and gradient projection method are reviewed (for more details see [4,8,9,15,16]).

From numerical point of view, many approaches for the obstacle problem and variational inequality have been suggested. The penalty method and regularization are popular for application. As we will see in Section 3.1, the bound constrained minimization problem (8) is equivalent to Linear Complementarity Problem (LCP). There are various methods for solving the following Linear Complementarity Problem

$$Ax - b = r, \quad x \geqslant 0, \quad r \geqslant 0, \quad x^T r = 0.$$

Several iterative algorithms have been developed for solving LCP problems. For example the principal pivoting algorithm of Cottle and Dantzig and complementarity pivoting algorithm by Lemak are well known (see [7]). The main idea in these algorithms is to reduce the LCP to the solution of a sequence of systems of linear equations in a way which is similar to the simplex method in linear programming.

In [13] an iterative primal–dual algorithm was suggested by Kunisch and Rendl which obtains the first order optimality and complementarity conditions related to (8), the feasibility is enforced by the update of the active set (see also [12]). Another alternative for bound constraint minimization is the interior point method. This method was initiated by Karmarkar for Linear Programming and extended to general case by Nesterov and Nemirovsky in 1988. In this method one minimizes the sequence of the parameterized barrier functions with Newton's method. As one of the advantages, the interior point methods is not sensitive to the conditioning of the Hessian matrix A. For more details about interior point methods, see the paper by Forsgren et al. [10].

The structure of this paper is as follows. First, we reformulate the problem as bound constraint quadratic minimization problem. Then, the methods that implement and combine the conjugate gradient method and its variants are briefly explained. Finally, numerical experiments are presented.

2. Reformulation of the problem

Problem (1) is equivalent to minimizing the functional

$$I = \int_\Omega \sum_{i=1}^m \left(\frac{1}{2}|\nabla u_i|^2 + f_i u_i\right) dx \tag{2}$$

over the set $\{(u_1,\ldots,u_m) | u_i - g_i \in H_0^1(\Omega), u_1 \geqslant u_2 \geqslant \cdots \geqslant u_m\}$. In [2] the system of coupled equations for the m-membrane problem is considered. For instance, when $m = 3$ and with given boundary conditions $g_1, g_2, g_3 (g_1 \geqslant g_2 \geqslant g_3)$, the Euler–Lagrange equation corresponding to the minimizer (u_1, u_2, u_3) is (see [2]):

$$\begin{aligned}
&\Delta u_1 = f_1 + \frac{f_2 - f_1}{2}\chi_{\{u_1=u_2\}} + \frac{2f_3 - f_2 - f_1}{6}\chi_{\{u_1=u_2=u_3\}} \quad \text{in } \Omega,\\
&\Delta u_2 = f_2 + \frac{f_1 - f_2}{2}\chi_{\{u_1=u_2\}} + \frac{f_3 - f_2}{2}\chi_{\{u_3=u_2\}} + \frac{2f_2 - f_1 - f_3}{6}\chi_{\{u_1=u_2=u_3\}} \quad \text{in} \quad \Omega,\\
&\Delta u_3 = f_3 + \frac{f_2 - f_3}{2}\chi_{\{u_2=u_3\}} + \frac{2f_1 - f_2 - f_3}{6}\chi_{\{u_1=u_2=u_3\}} \quad \text{in } \Omega,\\
&u_i = g_i \quad \text{on } \partial\Omega.
\end{aligned} \tag{3}$$