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Enhanced preliminary group classification of a class of generalized diffusion equations

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ABSTRACT

The method of preliminary group classification is rigorously defined, enhanced and related to the theory of group classification of differential equations. Typical weaknesses in papers on this method are discussed and strategies to overcome them are presented. The preliminary group classification of the class of generalized diffusion equations of the form $u_t = f(x,u)u_x^2 + g(x,u)u_{xx}$ is carried out. This includes a justification for applying this method to the given class, the simultaneous computation of the equivalence algebra and equivalence (pseudo) group, as well as the classification of inequivalent appropriate subalgebras of the whole infinite-dimensional equivalence algebra. The extensions of the kernel algebra, which are induced by such subalgebras, are exhaustively described. These results improve those recently published in *Commun Nonlinear Sci Numer Simul.*

1. Introduction

Group classification of differential equations is an efficient tool for investigating symmetry properties of classes of differential equations. These are differential equations that include arbitrary constants or functions of the independent and dependent variables as well as of derivatives of the dependent variables up to a certain order. It is known for a long time that depending on the value of these arbitrary elements the resulting differential equations from the given class can have different Lie invariance groups. The first examples of group classification were presented by Sophus Lie for the class of second order linear partial differential equations [16] and the class of second order ordinary differential equations [17]. Later, Ovsiannikov began the rigorous development of the theory of group classification [29]. In short, the solution of the group classification problem consists in finding the kernel of Lie invariance groups (i.e., those Lie symmetries that are admitted for all values of the arbitrary elements) and all inequivalent extensions of Lie invariance groups with respect to the kernel group. The equivalence involved means the similarity of equations up to transformations from a certain equivalence group (e.g., usual, generalized or conditional equivalence), see [33] for more detailed information.

For classes of differential equations being of simple structure (e.g., ones parameterized only by constants or functions of the same single argument), the corresponding group classification problems can be completely solved via compatibility analysis and explicit integration of the determining equations for Lie symmetries depending on values of the arbitrary elements and up to the equivalence chosen. Complete group classification can also be carried out for classes of differential equations possessing the normalization property. The algebraic method of classification effectively works for such classes. See the next section and also [33,38] for a more comprehensive review on different methods of group classifications.

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In the situation where the class depends in a more complicated way on its arbitrary elements, it may happen that both the determining equations are too difficult to be directly solved and the application of the algebraic method does not give the exhaustive solution. In this case, however, at least a partial solution of the group classification problem, known as *preliminary group classification*, is possible. The basic idea of preliminary group classification is to study only those extensions of the kernel group that are induced by the transformations from the corresponding equivalence group. The problem of finding inequivalent cases of such Lie symmetry extensions then reduces to the classification of inequivalent subgroups (resp. algebras) of the equivalence group (resp. algebra). This approach was first described in [1] and became prominent due to the paper [12].

Despite the approach of preliminary group classification is rather common, it is not well developed up to now. The basic mechanisms were formulated in [12] as two propositions for the specific class of nonlinear wave equations of the form $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$ and were later adopted in other papers for respective classes of equations. In the present paper, we state a stronger version of these propositions for general classes of differential equations. Another weakness commonly observed is that, when the equivalence algebra $\mathfrak{g}^{\sim}$ of the class of equations under consideration is infinite dimensional, only Lie symmetry extensions induced by subalgebras of a finite-dimensional subalgebra $\mathfrak{g}_0^{\sim}$ of $\mathfrak{g}^{\sim}$ are investigated, without giving any sound justification for the choice of $\mathfrak{g}_0^{\sim}$. In fact, this restriction is needless as it is possible to classify subalgebras of infinite-dimensional algebras in much the same way as subalgebras of finite-dimensional algebras [3–6,10,15,19,32,33,39]. It can even be simpler to classify low-dimensional subalgebras of the whole infinite-dimensional equivalence algebra $\mathfrak{g}^{\sim}$ as the adjoint action related to $\mathfrak{g}^{\sim}$ is more powerful and allows for greater simplification than the adjoint action corresponding to the finite-dimensional subalgebra $\mathfrak{g}_0^{\sim}$. One more common weakness in papers on the subject is that usually only extensions induced by one-dimensional subalgebras of equivalence algebras are studied. Moreover, these one-dimensional subalgebras (of a finite-dimensional subalgebra $\mathfrak{g}_0^{\sim}$ of $\mathfrak{g}^{\sim}$) are classified only with respect to the restricted equivalence relation which is generated by the adjoint representation of $\mathfrak{g}_0^{\sim}$. This leads to an overly large number of inequivalent subalgebras compared to the list of one-dimensional subalgebras that would be obtainable if the classification was done using the adjoint representation of the entire $\mathfrak{g}^{\sim}$.

In the present paper, we comprehensively carry out preliminary group classification for the class of (1 + 1)-dimensional second order quasilinear evolution equations of the general form

$$\Delta = u_t - f(x,u)u_x^2 - g(x,u)u_{xx} = 0, \tag{1}$$

where f and g are arbitrary smooth functions of x and u, and $g \neq 0$. The class (1) was considered in the recent paper [21] but results obtained therein are not correct. It is reviewed above that there are a number of typical weaknesses in papers on preliminary group classification, and results of [21] properly illustrate these weaknesses, cf. the first paragraphs of Sections 4 and 6 and Remark 8. This is why we aim to accurately present the revised preliminary group classification of the class (1) and to give all calculations in considerable detail.

The class (1) was considered in [21] as a class of generalized Burgers equations as it includes the *potential Burgers equation* as a particular element for the choice $f = g = 1$. This class also contains (1 + 1)-dimensional linear evolution equations, which correspond to the values $f = 0$ and g not depending on u. As a prominent example for a linear differential equation, one can recover the linear heat equation by choosing $f = 0$ and $g = 1$. An important subclass of the class (1) is the class of (1 + 1)-dimensional nonlinear diffusion equations of the general form $u_t = (F(u)u_x)_x$, where $F \neq 0$. It is singled from the class (1) by the constraints $g_x = 0$ and $f = g_u$. Moreover, any equation of the form (1) with $f_x = g_x = 0$ is reduced to a diffusion equation by a simple point transformation acting only on the dependent variable u. The solution of the group classification problem for this class by Ovsiannikov [28] (see also [1,29]) gave rise to the development of modern group analysis.

The class (1) is included in the wider class of equations $u_t = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x)$, for which the complete group classification was carried out in [3] by a method similar to that applied in the present paper. The fact that this class is normalized (cf. Section 2) played a crucial role in the entire consideration in [3]. However, as this class is essentially wider than the class (1), the corresponding equivalence algebras are rather different. This is why the results of [3] cannot be directly used for deriving the group classification of the class (1).

It should also be stressed that it is not natural to exclude linear differential equations from the present consideration. In fact, there are equations in the class (1) which are linearized by point transformations from the equivalence group $G^{\sim}$ of this class. The most prominent example of such a transformation in the above class is the transformation of the potential Burgers equation to the linear heat equation by means of the point transformation $\tilde{u} = e^u$ [26, p. 122]. That is, u is a solution of the potential Burgers equation whenever $\tilde{u}$ is a solution of the linear heat equation. In the course of preliminary group classification of the class (1) we encounter other examples of linearizable equations. Furthermore, the equivalence algebra $\mathfrak{g}_0^{\sim}$ of the subclass of (1), which is compliment to the subclass of linear equations and, therefore, singled out by the constraint $f^2 + {g_u}^2 \neq 0$, is much narrower than the equivalence algebra $\mathfrak{g}^{\sim}$ of the entire class (1). More precisely, the algebra $\mathfrak{g}_0^{\sim}$ is singled out as a subalgebra of $\mathfrak{g}^{\sim}$ by the constraint $h_{uu} = 0$, cf. Theorem 1.

The further organization of this paper is the following. The subsequent Section 2 discusses the theory of preliminary group classification. We generalize and extend assertions presented in [12] and formulate them rigorously using the modern language of group analysis. In Section 3 we derive the determining equations for Lie point symmetries of equations from the class (1) and find the corresponding kernel of Lie invariance algebras. The equivalence algebra $\mathfrak{g}^{\sim}$ and the equivalence group $G^{\sim}$ of the class (1) is computed in Sections 4 and 5, respectively. Throughout the paper, by the equivalence group we mean