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Representation of exact solution for the time-fractional telegraph equation in the reproducing kernel space

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ABSTRACT

The reproducing kernel theorem is used to solve the time-fractional telegraph equation with Robin boundary value conditions. The time-fractional derivative is considered in the Caputo sense. We discuss and derive the exact solution in the form of series with easily computable terms in the reproducing kernel space.

1. Introduction

In recent years, there are many researchers to develop numerical methods [1–6,16,17] for fractional partial differential equations due to the important application for fractional partial differential equation in fields of science and engineering [7–11].

The telegraph equation of hyperbolic equations is proved to be better model the suspension flows [12,13]. The time-fractional telegraph equations have recently been considered by many authors. Orsingher and Beghin [14] studied the fundamental solutions to time-fractional telegraph equations of order 2α. They obtained the Fourier transforms of the solutions for any α and gave a representation of their inverses in terms of stable densities. For the special case $\alpha = 1/2$, they also showed that the fundamental solution is the probability density of a telegraph process with Brownian time. Beghin and Orsingher [15] considered the fractional telegraph equation with partial fractional derivatives of rational order $\alpha = m/n$ with $m < n$. They proved that the fundamental solution to the Cauchy problem for the time-fractional telegraph equation can be expressed as the density of the composition of two processes, one depending on m and the other depending on n. Recently, Liu and coworkers [16] discuss and derive the analytical solution of the time-fractional telegraph equation with three kinds of nonhomogeneous boundary conditions by the method of separating variables, Momani [17] derived the analytic and approximate solutions of the space- and time-fractional telegraph equation with some special initial and boundary conditions using Adomian decomposition.

In this paper, we consider the representation of exact solution for time-fractional telegraph equation:

$$\begin{aligned} &D_t^{2\alpha}u(x,t) + aD_t^{\alpha}u(x,t) = k\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\ &0 < t \leqslant T, \quad 0 < x < L, \quad \frac{1}{2} < \alpha \leqslant 1, \end{aligned} \tag{1}$$

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subject to initial conditions

$$u(x,0)=\phi_1(x),\quad u_t(x,0)=\phi_2(x),\quad 0\leqslant x\leqslant L, \tag{2}$$

and boundary conditions

$$\begin{aligned} &u(0,t)+\lambda_1 u_x(0,t)=\mu_1(t),\quad 0\leqslant t\leqslant T,\\ &u(L,t)+\lambda_2 u_x(L,t)=\mu_2(t),\quad 0\leqslant t\leqslant T, \end{aligned} \tag{3}$$

where $D_t^{2\alpha}$ and D_t^{α} are Caputo fractional derivatives operator with respect to t, the rate a is an arbitrary nonnegative constant and k is an arbitrary positive constant, x and t are the space and time variables, f, ϕ_i, $\mu_i(i=1,2)$ are sufficiently smooth prescribed functions and $\lambda_i(i=1,2)$ are given prescribed constants.

The Caputo fractional derivative of order $\alpha>0$ is defined in [18] as

$$D_t^{\alpha}u(x,t)=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{\partial_\eta u(x,\eta)}{(t-\eta)^{1+\alpha-m}}d\eta, & m-1<\alpha<m,\\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha=m\in\mathbb{N}.\end{cases} \tag{4}$$

The main result is provided in Section 3, applying reproducing kernel theorem, we give the exact solution of problems (1)–(3) in the form of series in the reproducing kernel space.

2. Reproducing kernel space method

In order to solve Eq. (1) in reproducing kernel space, we need to transform the nonhomogeneous boundary conditions (2) and (3) into homogeneous boundary conditions, for the convenience, we still denote the solution of the new equation by $u(x,t)$, let

$$\mathbb{L}u(x,t)\triangleq D_t^{2\alpha}u(x,t)+aD_t^{\alpha}u(x,t)-k\frac{\partial^2 u(x,t)}{\partial x^2}=F(x,t), \tag{5}$$

subject to the initial and boundary conditions

$$\begin{aligned} &u(x,0)=0,\quad u_t(x,0)=0,\\ &u(0,t)+\lambda_1 u_x(0,t)=0,\\ &u(L,t)+\lambda_2 u_x(L,t)=0, \end{aligned} \tag{6}$$

where $\mathbb{L}:W(\Omega)\to L^2(\Omega), \Omega=[0,T]\times[0,L]$. In the following, we define the reproducing kernel space $W(\Omega)$.

Definition 2.1. The inner space $W(\Omega)=\{u(x,t)|\frac{\partial^4 u}{\partial x^2\partial t^2}$ is a completely continuous real value function in $\Omega, u(x,0)=0, u_t(x,0)=0, u(0,t)+\lambda_1 u_x(0,t)=0, u(L,t)+\lambda_2 u_x(L,t)=0, \frac{\partial^6 u}{\partial x^3\partial t^3}\in L^2(\Omega)\}$, where λ_i $(i=1,2)$ are given in (3). The inner product and norm in $W(\Omega)$ are given respectively by

$$\begin{aligned}\langle u(x,t),v(x,t)\rangle_w=&\sum_{i=0}^{2}\int_0^T\frac{\partial^{3+i}}{\partial t^3\partial x^i}u(0,t)\frac{\partial^{3+i}}{\partial t^3\partial x^i}v(0,t)\,dt+\int_0^T\int_0^L\frac{\partial^6}{\partial x^3\partial t^3}u(x,t)\frac{\partial^6}{\partial x^3\partial t^3}v(x,t)\,dx\,dt\\ &+\sum_{i=0}^{2}\frac{\partial^{2+i}}{\partial t^2\partial x^i}u(0,0)\frac{\partial^{2+i}}{\partial t^2\partial x^i}v(0,0)+\int_0^L\frac{\partial^5}{\partial t^2\partial x^3}u(x,0)\frac{\partial^5}{\partial t^2\partial x^3}v(x,0)\,dx,\ \|u\|_w\\ =&\sqrt{\langle u(x,t),u(x,t)\rangle_w}.\end{aligned} \tag{7}$$

Lemma 2.1. *$W(\Omega)$ is a reproducing kernel space. Its reproducing kernel function is*

$$R(x,y,t,s)=R1(x,y)R2(t,s), \tag{8}$$

where $R1(x,y)$ and $R2(t,s)$ are reproducing kernel functions of $W_1[0,L]$ and $W_2[0,T]$, respectively. For any $u(x,t)\in W(\Omega)$,

$$u(y,s)=(u(x,t),R(x,y,t,s))_w. \tag{9}$$