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Dynamics in the Kuramoto model with a bi-harmonic coupling function

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ABSTRACT

We study a variant of the Kuramoto model with a bi-harmonic coupling function, in which oscillators with positive first harmonic coupling strength are conformists and oscillators with negative first harmonic coupling strength are contrarians. We show that the model displays different synchronous dynamics and different dynamics may be characterized by the phase distributions of oscillators. There exist stationary synchronous states, travelling wave states, π state and, most interestingly, another type of nonstationary state: an oscillating π state. The phase distribution oscillates in a confined region and the phase difference between conformists and contrarians oscillates around π with a constant amplitude and a constant period in oscillating π state. Finally, the bifurcation diagram of the model in the parameter space is presented.

1. Introduction

The Kuramoto model of coupled phase oscillators has played a central role in the study of diverse systems in physics, biology and other domains since it was proposed by Kuramoto in 1975 [1], particularly those involving synchronization transition. Examples include the synchronous flashing of groups of fireflies [2,3], the coupling of oscillatory neurons in the suprachiasmatic nucleus of the brain governing circadian rhythms [4], the interaction of cells containing oscillatory chemically reacting constituents [5], Josephson junction circuits [6], applauding persons in a large audience [7], pedestrians on footbridges [8], and many other systems [9–15].

The original Kuramoto model consists of N phase oscillators. Each oscillator has its own natural frequency ω chosen from a given probability density $g(\omega)$ and interacts with the mean field with a global coupling strength K, which is positive, corresponding to an attractive interaction. A natural generalization of Kuramoto model is to allow K to have either sign. The negative coupling strength accounting for a repulsive interaction has been taken into consideration in recent years. Tsimring et al. considered the case in which the interaction between oscillators and the mean field is a repulsive one and found that phase locking among oscillators is destroyed for an array of non-identical phase oscillators provided that the number of oscillators is sufficiently large [16]. Some authors consider the local interaction among oscillators and found evidence of glassy behaviors when both positive and negative coupling strengths were allowed simultaneously [17,18]. Hong and Strogatz studied the situation in which the coupling strength is regarded as an oscillator's ability reacting to the mean

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field individually [19,20]. In their works, both positive and negative coupling strengths are present in the population. They found a surprising time-dependent state, a travelling wave state in which the mean field oscillates at a frequency different from the population's mean natural frequency and the phase difference between conformists and contrarians are locked at an angle away from π. Moreover, the globally coupled oscillator systems with two different kinds of oscillators have also been well examined in many other studies [21–23].

Kuramoto showed [24] that the interaction between phase oscillators should take the general form of $\Gamma(\phi_i - \phi_j)$, where ϕ_i and ϕ_j are the phases of oscillators i and j and Γ is a 2π-periodic function. The most of works on Kuramoto model only involve with the interaction taking the coupling form $\Gamma(\phi_i - \phi_j) = sin(\phi_i - \phi_j)$, which is the first harmonic of Γ in a Fourier expansion. Engelbrecht and Mirollo showed that the long-term average frequency as a function of the natural frequency displays a devil staircase when a second harmonic interaction term is introduced to the original Kuramoto model [25]. Tanaka and Aoyagi explored the behavior of phase oscillators with three-body interactions which actually leads to a second harmonic interaction term [26], they found that this system can take an infinite number of synchronized states in a structurally stable manner by varying the initial condition. Using a variation of recent dimensionality-reduction technique of Ott and Antonsen (OA) ansatz [27], Skardal et. al. studied coupled phase oscillators with single higher-order coupling [28] and characterized the cluster synchrony in the system. Recently, Komarov and Pikovsky study the Kuramoto model of globally coupled oscillators with a biharmonic coupling function [29], they provide an analytic self-consistency approach to find stationary synchronous states in the thermodynamic limit and demonstrate that there is a huge multiplicity of synchronous states. Li et al. investigate the Kuramoto model incorporated with the first harmonic and the second harmonic interaction terms [30], they show that the model displays the coexistence of multistable attractors and different attractors are characterized by the phase distributions of oscillators.

Next, we give two examples of realistic physical systems where the second harmonic term is strong or even dominating. The first example is recently experimentally realized φ Josephson junctions [31], where in the nonzero voltage state the phase "moves" viscously along a tilted periodic double-well potential. Therefore, one can expect strong effects caused by the second harmonics in the interaction. Another example is experiments with globally coupled electrochemical oscillators [32], a strong second harmonic components has been observed in the coupling function inferred from experimental data.

In this work, we will investigate the dynamics of the generalized Kuramoto model with a bi-harmonic interaction term. Here we focus on the model with a uniform probability density of natural frequency, which cannot be solved analytically by either the OA ansatz or the theoretical method proposed by Watanabe and Strogatz [33].

2. Model

The governing equations for the model are

$$\dot{\phi}_i = \omega_i + \frac{K_i}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i) + \frac{K}{N}\sum_{j=1}^{N}\sin(2(\phi_j - \phi_i)), i = 1, 2, \ldots, N. \tag{1}$$

where ϕ_i is the phase of the ith oscillator at time t and N is the number of phase oscillators in the system. ω_i is the natural frequency of the ith oscillator and is chosen at random from $[-\gamma, \gamma]$, where γ is the width of natural frequency distribution. K_i is the first harmonic coupling strength of the ith oscillator to the mean field and is chosen from a double-δ probability density $\Gamma(K) = (1-p)\delta(K-K_-) + p\delta(K-K_+)$, where $K_- < 0$ and $K_+ > 0$ represent the couplings for the contrarians and conformists, respectively, and p denotes the probability that a random oscillator is a conformist. K is the coupling strength for the interaction through the second harmonic term.

The collective rhythm in the model is quantified by a mean field-like quantity, namely, a complex order parameter $Re^{i\Phi}$ which is defined as

$$Z_m = R_m e^{im\Phi_m} = \frac{1}{N}\sum_{j=1}^{N} e^{im\phi_j}, m = 1, 2. \tag{2}$$

In the model (1), the extent of the synchronization is better reflected by the order parameter Z_2. However, as shown in the following, though the order parameter Z_1 is not responsible for the onset of synchronization, it is still an important measure reflecting the organization of oscillators in a microscopic view. R_1 and R_2 are the amplitudes of the order parameters Z_1 and Z_2, respectively, and Φ_1 and Φ_2 are corresponding average phases.

The complex order parameters in conformists and in contrarians are also important quantities to determine the dynamics in Eq. (1) and they are defined as $Z_\pm = R_\pm e^{i\Phi_\pm} = \frac{1}{N_\pm}\sum_{j\in S_\pm} e^{i\phi_j}$, where S_+ (or S_-) means the set of conformists (or contrarians) and $N_\pm$ are the numbers of conformists and contrarians, respectively.

3. Results and analysis

We numerically investigate the dynamics in Eq. (1) by a fourth-order Runge–Kutta algorithm with a time step $\delta t = 0.01$ and the quantities of interest are measured after a sufficient long transient is discarded. Throughout the work, we let $N = 10000$, $K_- = -1.0$, $K_+ = 2.0$, $K = 0.1$ and $\gamma = 0.1$ unless specified. Initially, we assign each oscillator a phase randomly drawn from [0, 2π].