Commun Nonlinear Sci Numer Simulat 38 (2016) 91–101

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Semi-analytical expression of stochastic closed curve attractors in nonlinear dynamical systems under weak noise

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ARTICLE INFO

Article history:
Received 18 December 2015
Revised 2 February 2016
Accepted 4 February 2016
Available online 23 February 2016

Keywords:
Stochastic attractors
Probability density function
Stochastic sensitivity function
Closed curve attractors

ABSTRACT

In this paper, a simple but accurate semi-analytical method to approximate probability density function of stochastic closed curve attractors is proposed. The expression of distribution applies to systems with strong nonlinearities, while only weak noise condition is needed. With the understanding that additive noise does not change the longitudinal distribution of the attractors, the high-dimensional probability density distribution is decomposed into two low-dimensional distributions: the longitudinal and the transverse probability density distributions. The longitudinal distribution can be calculated from the deterministic systems, while the probability density in the transverse direction of the curve can be approximated by the stochastic sensitivity function method. The effectiveness of this approach is verified by comparing the expression of distribution with the results of Monte Carlo numerical simulations in several planar systems.

1. Introduction

Noise or fluctuation is inevitably in dynamical systems models like mechanical, lasers, electronic circuits, chemical and biological systems which often disturbed by weak noises either due to reduction of system variables (internal noise) or due to environmental influence (external noise). For nonlinear dynamical systems, the interaction between nonlinearity and randomness can induce many nontrivial phenomena which have no analogue in the deterministic cases. Until now various noise-induced behaviors have been found, such as stochastic resonance [1], coherence resonance [2], noise-induced synchronization [3], noise-induce bifurcation [4] and noise-induced chaos [5], etc.

When an attractor is disturbed by noises, the trajectories will leave this attractor temporarily, but most of them can be attracted back and form a cloud or bundle around the deterministic attractor, which is called stochastic attractor [6]. Under weak noises, when the initial probability density function (PDF) concentrates in the original deterministic attractor, the state of the system can stay at stochastic attractor for a very long time before the trajectory escapes to other deterministic invariant sets. Thereby, for weak noises perturbed systems, the response can be seen as stochastic attractors around the deterministic attractors and transitions between them. Thus it is meaningful to reveal PDF of stochastic attractors because the systems states are under them almost all the time. Although numerical simulation can easily describe the distribution of stochastic attractors, analytical methods are needed to both save the computation time and reveal the relation between distribution characters and system parameters. For continuous-time dynamical systems, the evolution of PDF can be detailed

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http://dx.doi.org/10.1016/j.cnsns.2016.02.006
1007-5704/

by Fokker–Planck–Kolmogorov equation while for discrete-time dynamical systems by Frobenius–Perron integral equation. But often it is a difficult task to use these equations directly when dimension of the system is larger than one, thus many approximate methods were raised to obtain PDF under various specific conditions [7–11], especially in nonlinear vibration field in which case the nonlinearity is weak and the solutions depend on small parameters [12].

Closed curve attractor is the simplest nontrivial attractor in nonlinear dynamical systems. It can be periodic or quasi-periodic (in discrete-time systems). Earlier researches of noise-disturbed closed curve attractors concentrated on limit cycle attractors near Andronov–Hopf bifurcation points, in which case the radial motion and angular motion can be separated [13,14]. For a fully developed limit cycle which is far from bifurcation points, Dykman [15] proposed a complex formula to get the stationary probability distribution near the circles. For orbital diffusions of closed curve, on the basis of the quasi-potential theory, Ryashko [16] proposed a method to analysis the stability of limit cycles in nonlinear dynamical systems. This idea was then further developed by Bashkirtseva who proposed the concept of stochastic sensitivity function (SSF) to describe the dispersion of stochastic closed curve attractors in the transverse direction [17,18]. The SSF method sheds light on the construction of PDF of stochastic non-trivial attractors in dynamical systems, especially for closed curve attractors. In this paper, it is indicated that, for closed curve attractors in weak additive noise disturbed system, the n-dimensional PDF of stochastic attractors can be decomposed into 1-dimensional longitudinal PDF and $(n-1)$-dimensional transverse PDF. The former is actually the natural measure of the deterministic circle and the latter can be calculated by SSF method. Consequently, the n-dimensional PDF of stochastic attractors can be expressed by production of 1-dimensional longitudinal PDF and $(n-1)$-dimensional transverse PDF.

The outline of the paper is as follows: In Section 2, the approximate analytical expression of stochastic closed curve attractors is presented. In order to get the transverse distribution, SSF method of limit cycle is detailed in Section 3 while in Section 4 SSF method for quasi-periodic closed curve attractors in discrete-time systems is discussed. In Section 5 validity of this approximate method is verified by several kinds of planar closed curves. Conclusions are drawn in Section 6.

2. Approximation formula of stochastic closed curve attractor

Consider a closed curve attractor Γ in nonlinear dynamical system. According to Tél [19], the probability that the trajectory visits different parts of the attractor (longitudinal PDF) is independent of noise intensity when noise is additive. That is to say, the 1-dimensional PDF of the disturbed circle coincides with the deterministic nature measure. So if we only focus on the orbital character of circle Γ(phase character is ignored), it can be regarded that additive noise just pushes points away from the attractor transversely because the local probability flux in the longitudinal direction reaches a dynamical balance.

When noise is weak and Gaussian, points deviate from the closed curve attractor and form a Gaussian distribution in the orthogonal hyperplane. It is found that although the Gaussian distributions in different parts of the attractor can intersect each other [20], which destroys the invariance of longitudinal distribution and Gaussian shape of orthogonal distribution, this kind of intersect only happens in the tails of the Gaussian distributions and can be neglected for weak noise. Thus it is can be concluded that for weak additive Gaussian white noise disturbed closed curve attractor; (1) The PDF of stochastic attractors in the longitudinal direction equals the deterministic nature measure of the attractor; (2) The transverse PDF in the orthogonal hyperplane satisfies Gaussian distribution; (3) The longitudinal and the transverse PDFs are independent with each other. These conclusions coincide with the foundation in [15] that the product of the velocity along the circle times the area of the cross section of the probability distribution transverse to the circle is constant.

Based on above understanding, a semi-analytical expression of stochastic closed curve attractors is presented which takes a simpler form compared with formula in [15]. Two types of coordinates need to be introduced. One is the global longitudinal coordinate s along the curve, and the other is local $(n-1)$ dimensional coordinate vector $\boldsymbol{z}$ which is defined in the transverse hyperplane at every point on the circle and originated at this point (see Fig. 1 for planar closed curve). The n-dimensional PDF of stochastic attractors can now be expressed by production of two independent low-dimensional PDF. One is 1-dimensional PDF, ρ_τ, that describes PDF along the global longitudinal coordinates, and the other is $(n-1)$-dimensional PDF, ρ_n, which takes a Gaussian form and expresses PDF in the orthogonal hyperplane, namely:

$$\rho(\boldsymbol{x}) = \rho_\tau(s(\boldsymbol{x}^*))\rho_n(\boldsymbol{z}(\boldsymbol{x})) \tag{1}$$

where $\boldsymbol{x}^*$ is the point on the circle that is nearest to $\boldsymbol{x}$.

Because noise cannot change the longitudinal PDF of stochastic attractors, this 1-dimenional distribution can be thus calculated from the deterministic attractors. First consider limit cycles in continuous-time systems. Suppose a point $\boldsymbol{x}^*$ on the limit cycle Γ and an arc-length Δs which contains $\boldsymbol{x}^*$.The magnitude of velocity at point $\boldsymbol{x}^*$ is**:**

$$\upsilon(\boldsymbol{x}^*) = \|\boldsymbol{f}(\boldsymbol{x}^*)\| \tag{2}$$

where $\|\bullet\|$ represents 2-norm.When $\Delta s \to 0$, the time for the trajectory to travel over this arc-length can be expressed as:

$$dt = \lim_{\Delta s\to 0}\frac{\Delta s}{\upsilon(\boldsymbol{x}^*)} = \frac{ds}{\upsilon(\boldsymbol{x}^*)} \tag{3}$$

Thereby, the probability of finding the state in this arc-length is:

$$P(s(\boldsymbol{x}^*)) = \frac{dt}{T} = \frac{ds}{\upsilon(\boldsymbol{x}^*)T} \tag{4}$$