Commun Nonlinear Sci Numer Simulat 38 (2016) 243–256

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Normal and quasinormal forms for systems of difference and differential-difference equations

Ilya Kashchenko [a], Sergey Kaschenko [a,b,*]

[a] Yaroslavl State University, Sovetskaya 14, Yaroslavl, Russian Federation 150000, Russia
[b] National Research Nuclear University MEPhI, Kashirskoye shosse 31, Moscow, Russian Federation 115409, Russia

ARTICLE INFO

Article history:
Received 2 April 2014
Revised 2 February 2016
Accepted 27 February 2016
Available online 4 March 2016

Keywords:
Normal form
Quasinormal form
Difference equation
Delay differential equation

ABSTRACT

The local dynamics of systems of difference and singularly perturbed differential-difference equations is studied in the neighborhood of a zero equilibrium state. Critical cases in the problem of stability of its state of equilibrium have infinite dimension. Special nonlinear evolution equations, which act as normal forms, are set up. It is shown that their dynamics defines the behavior of solutions to the initial system.

1. Introduction

The systems of nonlinear difference and differential-difference equations provide mathematical models for many applications [1–3], especially in lasers [4,5], medicine [6–9], neural networks [10] and information processing [11]. In this respect one can specially single out models for laser physics, neural networks, cellular automata, etc. Systems of the following form are one of the basic objects in the theory of systems of difference equations:

$$u_n = (A + \varepsilon B)u_{n-1} + F(u_{n-1}). \tag{1}$$

Here $n = 0, 1, \dots;$ $u_n \in R^r$, A and B are $r \times r$ matrices, $\varepsilon > 0$ is a small parameter: $0 < \varepsilon \ll 1$, the vector function $F(u)$ is sufficiently smooth and has an order of smallness not less than second in neighborhood of zero. It is convenient to suppose that $F(u) = F_2(u,u) + F_3(u,u,u) + \cdots$, where $F_j(u,\dots,u)$ are linear in each argument. We assume the critical case in the problem of stability of the zero state of equilibrium where the matrix A has m eigenvalues which are equal to 1 in absolute value, and its remaining eigenvalues are less than 1 in absolute value. It is well known that under such conditions the system (1) has in the neighborhood of the state of equilibrium $u_n \equiv 0$ a local invariant integral manifold of dimension m, on which (1) may be represented as a special system of dimension m of nonlinear equations, which is a normal form. We note that the general theory of one-dimensional mappings is well studied in [12]. We mention the special theory of piecewise-linear continuous mappings developed in [13], and that of piecewise-linear discontinuous mappings developed in [14]. There is no completed theory even for a two-dimensional case but many interesting results have been obtained for various nonlinearity classes (see, e.g., [15]).

* Corresponding author at: Yaroslavl State University, Sovetskaya 14, Yaroslavl, Russian Federation 150000, Russia. Tel.: +79106640443.
E-mail addresses: iliyask@uniyar.ac.ru (I. Kashchenko), kasch@uniyar.ac.ru (S. Kaschenko).

http://dx.doi.org/10.1016/j.cnsns.2016.02.041
1007-5704/

At the present time, a general research technique for the study of a local dynamics for systems of the form (1) in a sufficiently small neighborhood of zero equilibrium is well developed. Here, the results of [16] on the stability of zero equilibrium in various critical cases will be considerably used. The essence of these results is that normal forms for difference systems can be represented as special systems of ordinary differential equations for slowly varying (in continuous time t) amplitudes at the harmonics of the linear part (1).

Along with (1), we consider a system with continuous time in R^r

$$u(t) = (A + \varepsilon B)u(t-1) + F(u(t-1)). \tag{2}$$

This system of equations belongs to the class of systems of equations of neutral type of zero order. Its solutions with a piecewise continuous initial (at $t \in [t_0 - 1, t_0]$) vector function will also be piecewise continuous at $t > t_0$.

The continuity condition for solutions (2) with the initial function $u(s + t_0) = \varphi(s) \in C_{[-1,0]}(R^r)$ is to require that the following matching condition be satisfied:

$$\varphi(0) = (A + \varepsilon B)\varphi(-1) + F(\varphi(-1)).$$

The discussion here will focus on the issues of local dynamics of solutions to the system of Eq. (2) with piecewise continuous, and continuous solutions with matching conditions for the initial functions. Therefore, the steady–state solutions will either be piecewise continuous or, as $t \to \infty$, tend to piecewise continuous functions.

We note that a characteristic quasipolynomial of the linear part (2) at $\varepsilon = 0$ is of the form

$$\det |A - e^{\lambda} I| = 0. \tag{3}$$

Let $\varkappa_1, \dots, \varkappa_m$ denote all eigenvalues of the matrix A, which are equal to 1 in absolute value. Then $\varkappa_j = \exp(i\omega_j)$ $(0 \le \omega_j < 2\pi)$ and the roots of (3) are an infinite collection.

$$\lambda_{jk} = i\big[\omega_j + 2\pi k\big],\ j = 1, \dots, m;\ \ k = 0, \pm 1, \pm 2, \dots\ .$$

Thus, for (2), a critical case of infinite dimension is realized in the problem of stability of zero equilibrium. As a consequence, the presence of infinite sets of various local discontinuous dynamical conditions is characteristic of the system (2). In what follows, we consider a system of equations which is equivalent to the system (2).

$$u(t + \mu) = (A + \varepsilon B)u(t-1) + F(u(t-1)), \tag{4}$$

where

$$0 < \mu \ll 1 \tag{5}$$

is another small parameter. This system results from (2) by simple transformations and changes of notation:

$$v(s) = u\left(\frac{s}{1+\mu}\right), \quad s = (1+\mu)t - \mu,$$

followed by the replacement of $v(s)$ by $u(t)$.

The solution of (4) with the initial function $u(s) = \varphi(s) \in C_{[-1,\mu]}(R^r)$ will also be n-times continuously differentiable if the matching conditions are satisfied.

$$\frac{d^j\varphi(s)}{ds^j}\Big|_{s=\mu} = (A + \varepsilon B)\frac{d^j\varphi(s)}{ds^j}\Big|_{s=-1} + \frac{d^jF(\varphi(s))}{ds^j}\Big|_{s=-1} \quad (j = 0, \dots, n).$$

In addition to the problem of studying the dynamics of continuously differentiable and twice continuously differentiable solutions (2), there naturally arises the problem of studying the local dynamics of the following two systems of equations

$$\mu\dot{u} + u = (A + \varepsilon B)u(t-1) + F(u(t-1)) \tag{6}$$

and

$$\frac{1}{2}\mu^2\ddot{u} + \mu\dot{u} + u = (A + \varepsilon B)u(t-1) + F(u(t-1)). \tag{7}$$

Each of these systems is a system with a delayed argument, which means that no matching conditions of the initial functions are required for the smoothness of solutions (with all t larger than some t_0). By virtue of (5), the systems (6) and (7) are singularly perturbed and equivalent to systems with large delay. Systems of such kind may demonstrate complex dynamics [17–19]. Some methods for constructing exact solutions are studied in [20].

We formulate the problem of comparative local analysis of the dynamics of the systems (2), (6) and (7) (with ε and μ sufficiently small).

Characteristic quasipolynomials of the systems (2), (6) and (7), linearized in zero, have infinitely many roots, the real components of which vanish as $\varepsilon, \mu \to 0$. Thus, in the problem of stability, a critical case of infinite dimension is realized for these systems, too.

Standard methods of local analysis, which are based on the application of methods of invariant integral manifolds [21,22] and the method of normal forms (see, e.g., [23]), are, generally speaking, not applicable here but the formalism of the method of normal forms is considerably used. To study dynamics under the conditions formulated in [24], methods for the