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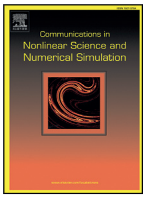

Bifurcation analysis of a discrete-time ratio-dependent predator–prey model with Allee Effect

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ABSTRACT

A discrete-time predator–prey model with Allee effect is investigated in this paper. We consider the strong and the weak Allee effect (the population growth rate is negative and positive at low population density, respectively). From the stability analysis and the bifurcation diagrams, we get that the model with Allee effect (strong or weak) growth function and the model with logistic growth function have somewhat similar bifurcation structures. If the predator growth rate is smaller than its death rate, two species cannot coexist due to having no interior fixed points. When the predator growth rate is greater than its death rate and other parameters are fixed, the model can have two interior fixed points. One is always unstable, and the stability of the other is determined by the integral step size, which decides the species coexistence or not in some extent. If we increase the value of the integral step size, then the bifurcated period doubled orbits or invariant circle orbits may arise. So the numbers of the prey and the predator deviate from one stable state and then circulate along the period orbits or quasi-period orbits. When the integral step size is increased to a critical value, chaotic orbits may appear with many uncertain period-windows, which means that the numbers of prey and predator will be chaotic. In terms of bifurcation diagrams and phase portraits, we know that the complexity degree of the model with strong Allee effect decreases, which is related to the fact that the persistence of species can be determined by the initial species densities.

1. Introduction

In the traditional mathematical ecology literature, the most classical predator–prey model is Gause type system with the following form [1]

$$\begin{cases} \dfrac{dx}{dt} = xg(x, K) - yp(x), \\ \dfrac{dy}{dt} = y(-d + cq(x)), \end{cases}$$

where $g(x, K)$ is the growth function of the prey in the absence of the predator. $p(x)$, $q(x)$ are the functional response and the growth function of the predator, respectively. K is the environmental carrying capacity of the prey. c and d are the growth rate and mortality rate of the predator, respectively.

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This is a system with a classical unique prey-dependent functional response. For this system most authors have considered the simplest case of the logistic prey growth function and Holling type functional response [2–7]. Recently, however, traditional prey-dependent models have been challenged by several ecologists, based on the fact that the functional responses ought to depend on the ratio of prey/predator or predator/prey. The authors [8] have concluded that the ratio-dependent functional response is a simple and suitable pattern to illustrate many types of heterogeneity that occur in large scale natural systems, while, in contrast, the prey-dependent functional responses may be more appropriate for homogeneous systems. Kuang and Beretta [9] have shown that compared to the traditional predator–prey systems, ratio-dependent models have much richer boundary dynamics and there can be no paradox of enrichment or paradox of biological control. Hence ratio-dependent systems have received much attention in recent years [10–13].

Allee effect [14] is a crucial phenomenon in the biological world. A population is said to have an Allee effect, if the growth rate per capita is initially an increasing function for the low density. Moreover it is called a strong Allee effect if the per capita growth rate in the limit of low density is negative, and a weak Allee effect means that the per capita growth rate is positive at zero density. A strong Allee effect introduces a population threshold [14], and the population must surpass this threshold to grow. In contrast, a population with a weak Allee effect does not have a threshold [15]. It has been shown that replacing the logistic growth function with Allee-type function could dramatically change the patterns of dynamics, thus there is an urgent need for mathematical analysis of the behavior of ratio-dependent prey–predator models with more realistic Allee effect growth functions. In [16–18], the authors have considered the continuous-time models with Allee effect, and several discrete-time models with Allee effect are also discussed in [19–22]. In [16], the authors have considered the following continuous-time model

$$\begin{cases} \dfrac{dx}{dt} = x(x-\beta)(1-x) - \frac{\alpha xy}{x+y}, \\ \frac{dy}{dt} = \dfrac{\alpha_1 xy}{x+y} - \delta y. \end{cases} \tag{1}$$

They have discussed saddle-node bifurcation, Hopf bifurcation, Bogdanov–Takens bifurcation and also analyzed the biological significance of the model with strong and weak Allee effect.

However, for a mathematical biology model, Freedman [1] points out that when the population numbers are small or births and deaths all occur at discrete times, or within certain intervals of time, the discrete-time model would indeed be more realistic than the continuous-time model. We also know that numerical solutions or approximate solutions of the discrete-time models can be obtained more easily and much work has shown that discrete-time models can produce much richer dynamical behaviors than continuous-time models [1,23–27].

Motivated by the above-mentioned works, in order to compare the continuous-time system with the discrete-time system and observe some dynamical behaviors that the continuous-time system does not have, we discuss the following corresponding discrete-time system

$$\begin{cases} x \to x + \mu\left[x(x-\beta)(1-x) - \dfrac{\alpha xy}{x+y}\right], \\ y \to y + \mu\left[\dfrac{\alpha_1 xy}{x+y} - \delta y\right], \end{cases} \tag{2}$$

where x, y denote the densities of the prey and the predator, respectively. δ, α_1 and α indicate the predator's death rate, growing ability and consumption ability, respectively. μ is the integral step size and $-1 < \beta \ll 1$ is the Allee effect parameter. $\beta > 0$ is the rescaled Allee threshold for x in the absence of the predator. In this paper, we discuss the stability of fixed points of model (2), then analyze three types of bifurcations and proceed some numerical simulations which verify our theoretical analysis. By analysis, we know that, compared with the continuous-time model, the discrete-time model has different stability properties and bifurcation structures.

The outline of this paper is as follows. In Section 2 we discuss the existence and local stability of fixed points of model (2). In Section 3 we study fold bifurcation by choosing δ as a bifurcation parameter, flip bifurcation and Hopf bifurcation by choosing μ as a bifurcation parameter. In Section 4 we present numerical simulations, which not only illustrate the results of theoretical analysis, but also exhibit complex and new dynamical behaviors. Finally, we conclude this paper with comments and discussions.

2. The existence and stability of fixed points

The fixed points of model (2) satisfy the following equations

$$\begin{cases} x + \mu x(x-\beta)(1-x) - \dfrac{\alpha\mu xy}{x+y} = x, \\ y + \dfrac{\alpha_1 \mu xy}{x+y} - \delta\mu y = y. \end{cases} \tag{3}$$

By simple analysis, we get the following existence proposition of the fixed points.