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Caputo derivatives of fractional variable order: Numerical approximations☆

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ABSTRACT

We present a new numerical tool to solve partial differential equations involving Caputo derivatives of fractional variable order. Three Caputo-type fractional operators are considered, and for each one of them an approximation formula is obtained in terms of standard (integer-order) derivatives only. Estimations for the error of the approximations are also provided. We then compare the numerical approximation of some test function with its exact fractional derivative. We end with an exemplification of how the presented methods can be used to solve partial fractional differential equations of variable order.

1. Introduction

As is well known, several physical phenomena are often better described by fractional derivatives [11,18,36]. This is mainly due to the fact that fractional operators take into consideration the evolution of the system, by taking the global correlation, and not only local characteristics. Moreover, integer-order calculus sometimes contradict the experimental results and therefore derivatives of fractional order may be more suitable [12].

An interesting recent generalization of the theory of fractional calculus consists to allow the fractional order of the derivative to be non-constant, depending on time [5,19,20]. With this approach, the non-local properties are more evident and numerous applications have been found in physics, control and signal processing [7,13,21,22,26,27,34]. One difficult issue, that usually arises when dealing with such fractional operators, is the extreme difficulty in solving analytically such problems [2,37]. Thus, in most cases, we do not know the exact solution for the problem and one needs to seek a numerical approximation. Several numerical

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methods can be found in the literature, typically applying some discretization over time or replacing the fractional operators by a proper decomposition [2,37].

Recently, new approximation formulas were given for fractional constant order operators, with the advantage that higher-order derivatives are not required to obtain a good accuracy of the method [1,23,24]. These decompositions only depend on integer-order derivatives, and by replacing the fractional operators that appear in the problem by them, one leaves the fractional context ending up in the presence of a standard problem, where numerous tools are available to solve them. Here we extend such decompositions to Caputo fractional problems of variable order.

The paper is organized as follows. To start, in Section 2 we formulate the needed definitions. Namely, we present three types of Caputo derivatives of variable fractional order. First, we consider one independent variable only (Section 2.1); then we generalize for several independent variables (Section 2.2). Section 3 is the main core of the paper: we prove approximation formulas for the given fractional operators and upper bound formulas for the errors. To test the efficiency of the proposed method, in Section 4 we compare the exact fractional derivative of some test function with the numerical approximations obtained from the decomposition formulas given in Section 3. To end, in Section 5 we apply our method to approximate two physical problems involving Caputo fractional operators of variable order (a time-fractional diffusion equation in Section 5.1 and a fractional Burgers' partial differential equation in fluid mechanics in Section 5.2) by classical problems that may be solved by well-known standard techniques.

2. Fractional calculus of variable order

In the literature of fractional calculus, several different definitions of derivatives are found [28]. One of those, introduced by Caputo in 1967 [3] and studied independently by other authors, like Džrbašjan and Nersesjan in 1968 [10] and Rabotnov in 1969 [25], has found many applications and seems to be more suitable to model physic phenomena [6,8,9,15,16,31,33,35]. Before generalizing the Caputo derivative for a variable order of differentiation, we recall two types of special functions: the Gamma and Psi functions. The Gamma function is an extension of the factorial function to real numbers, and is defined by

$$\Gamma(t) = \int_0^\infty \tau^{t-1} \exp(-\tau)\, d\tau, \quad t > 0.$$

We mention that other definitions exist for the Gamma function, and it is possible to define it for complex numbers, except the non-positive integers. A basic but fundamental property that we will use later is the following:

$$\Gamma(t+1) = t\,\Gamma(t).$$

The Psi function is the derivative of the logarithm of the Gamma function:

$$\Psi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}.$$

Given $\alpha \in (0,1)$, the left and right Caputo fractional derivatives of order α of a function $x : [a,b] \to \mathbb{R}$ are defined by

$${}_a^C D_t^\alpha x(t) = {}_a D_t^\alpha (x(t) - x(a))$$

and

$${}_t^C D_b^\alpha x(t) = {}_t D_b^\alpha (x(t) - x(b)),$$

respectively, where ${}_a D_t^\alpha x(t)$ and ${}_t D_b^\alpha x(t)$ denote the left and right Riemann–Liouville fractional derivative of order α, that is,

$${}_a D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} x(\tau) d\tau$$

and

$${}_t D_b^\alpha x(t) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha} x(\tau) d\tau.$$

If x is differentiable, then, integrating by parts, one can prove the following equivalent definitions:

$${}_a^C D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} x'(\tau) d\tau$$

and

$${}_t^C D_b^\alpha x(t) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} x'(\tau) d\tau.$$

From these definitions, it is clear that the Caputo fractional derivative of a constant is zero, which is false when we consider the Riemann–Liouville fractional derivative. Also, the boundary conditions that appear in the Laplace transform of the Caputo derivative depend on integer-order derivatives, and so coincide with the classical case.