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On one-step worst-case optimal trisection in univariate bi-objective Lipschitz optimization

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ABSTRACT

The bi-objective Lipschitz optimization with univariate objectives is considered. The concept of the tolerance of the lower Lipschitz bound over an interval is generalized to arbitrary subintervals of the search region. The one-step worst-case optimality of trisecting an interval with respect to the resulting tolerance is established. The theoretical investigation supports the previous usage of trisection in other algorithms. The trisection-based algorithm is introduced. Some numerical examples illustrating the performance of the algorithm are provided.

1. Introduction

In the field of multi-objective non-convex optimization prevail metaheuristic methods [1] when the potential of methods based on mathematical models is not fully exploited. A theoretical problem of special interest is the construction and investigation of the methods optimal with respect to the well substantiated mathematical models of non-convex problems. In the field of single-objective non-convex optimization both traditional paradigms, of worst case [2] and average case [3] optimality, were thoroughly investigated, and at least some of these results can be generalized to the multi-objective case. For example, the worst case optimal bi-objective algorithm for Lipschitz objective functions is shown in [4] coincident with the algorithm for covering a feasible region by balls of minimum radius; an analogous result for single objective case was proved considerably earlier in [5]. Similarly, the well known one-step optimal algorithm by Shubert–Pijavsky [6,7] was generalized for the bi-objective case in [8]. The applicability of optimal algorithms can be narrow either because of the difficulty to check the grounding assumptions in praxis or because of the complexity of the implementation of the optimal algorithms. However, the knowledge of properties of the optimal algorithms can be helpful in implementation of practically suitable algorithms which posses some of these properties [9–14].

In the present paper the bi-objective optimization for single variable Lipschitzian objectives is considered where a feasible interval is sequentially partitioned: a selected subinterval is partitioned by trisection, the worst case optimality of which is established. The favorable properties of a univariate trisection based single-objective global optimization algorithm were shown in [15]. The optimal trisection algorithm for univariate problems is also of interest for the development of multivariate algorithms by the diagonal scheme. For the details of such an approach in single objective multidimensional optimization we refer to [16–21]. The results of the present paper corroborate the trisection used in [22]. The considered trisection is related to the optimal partition of an interval. Optimal partition is also characteristic to other methods, the optimality of which is originally defined in

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other terms. To highlight the mentioned property, we start the next section with the respective interpretation of the well known method by Shubert–Pijavsky [6,7]. Next, an optimal version of trisection in the univariate single-objective Lipschitz optimization is considered. The main result is presented in the third section. Finally, several numerical examples are provided enabling the comparison of the bisection and trisection based algorithms.

2. One-step worst-case optimal methods for single-objective univariate optimization

2.1. Bisection in the single-objective case

Let us consider a univariate single-objective Lipschitz optimization problem

$$\min_{x\in A} f(x), \tag{1}$$

$$|f(x) - f(y)| \le L|x - y|,\ \ x, y \in A,\ \ L > 0, \tag{2}$$

where A is a closed interval, and L is the Lipschitz constant.

A theoretically substantiated [6,7] and intuitively perceptible algorithm, aimed at the solution of (1), combines the iterative update of the Lipschitzian lower bound for the objective function values $F_n(x)$, and computation of the current $(n+1)$th value of $f(x)$ at the minimum point of $F_n(x)$:

$$F_n(x) = \max_{i=1,\dots,n} (y_i - L|x - x_i|), \tag{3}$$

$$x_{n+1} = \arg\min_{x\in A} F_n(x), \tag{4}$$

where it is supposed that at n previous iterations the objective function values were computed at the points x_i, $i = 1, \dots, n$, and $y_i = f(x_i)$. This algorithm is frequently called the Shubert–Pijavsky algorithm according to the names of the authors of [6,7], and it is of interest to us, as seemingly the first, one-step worst-case optimal global optimization algorithm. Below we state the optimality conditions in a form suitable for the unified analysis of the other cases, e. g. trisection based algorithms.

The value $y_{on} = \min_{1\le i\le n} y_i$ is accepted as an estimate of the global minimum with respect to the information available after n function value computations. Such a decision seems sufficiently rational and does not require a further justification. Therefore the worst-case error of the estimate with respect to the available information is equal to

$$\Delta_n(X_n, Y_n) = \max_{f(\cdot)\in\Phi(L,n)} (y_{on} - \min_{x\in A} f(x)) \tag{5}$$

$$= y_{on} - \min_{x\in A} F_n(x), \tag{6}$$

where $\Phi(L, n)$ is the subclass of Lipschitz functions which satisfy (2) and

$$X_n = (x_1, \dots, x_n), Y_n = (f(x_1), \dots, f(x_n)). \tag{7}$$

The one-step optimality means that x_{n+1}, the site of the current computation of objective function value, should be selected to minimize the potential error after that computation:

$$x_{n+1} = \arg\min_{x\in A} \max_{f(\cdot)\in\Phi(L,n)} \Delta_{n+1}((X_n, x), (Y_n, f(x))). \tag{8}$$

It is easy to show that the formula (8) can be reduced to the following one:

$$x_{n+1} = \arg\min_{x\in A} F_n(x). \tag{9}$$

The function $F_n(x)$ is piecewise linear, as obvious from (3), and its global minimizer is computable using a simple analytical formula. The left side graph in Fig. 1 illustrates the definition of x_{n+1} in the selected subinterval of its location. In this figure, the shaded subinterval indicates the search region where smaller objective function values than the best one known possibly exist. The point x_{n+1} is the center of the considered subinterval, and its location is coincident with that defined by the optimal algorithm [5] with the budget of one computation of the objective function value. As it is proved in [5], the worst-case optimal algorithm for the search for global minimum of Lipschitz continuous functions (with the predefined number of computations of objective function values N) is coincident with the algorithm for selecting the centers of N balls of minimum radius to cover the feasible region. Similarly, a center of one ball is chosen in one-step optimal algorithm. In such a situation the feasible region for global minimizer is defined by the points α_1, α_2 (see left side of Fig. 1) where the lower Lipschitz bound intersects the horizontal line at the level y_{on}. The site for x_{n+1} coincides with the center of a ball/subinterval of the optimal cover of the subinterval [α_1, α_2] assuming that this subinterval should be covered by three balls/subintervals with the centers α_1, α_2 and x_{n+1}.

Note, that the other subintervals of A, besides of that containing (supposedly a single) global minimizer of $F_n(x)$, can include regions of possible improvement of y_{on}; but the selection of a site for the next computation of objective functions in such regions can not guarantee the reduction of the worst-case error.