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Short communication

Rotational and self-similar solutions for the compressible Euler equations in R^3

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ABSTRACT

In this paper, we present rotational and self-similar solutions for the compressible Euler equations in R^3 using the separation method. These solutions partly complement Yuen's irrotational and elliptic solutions in R^3 (Yuen, 2012) [17] as well as rotational and radial solutions in R^2 (Yuen, 2014) [18]. A newly deduced Emden dynamical system is obtained. Some blowup phenomena and global existences of the responding solutions can be determined. The 3D rotational solutions provide concrete reference examples for vortices in computational fluid dynamics.

1. Introduction

In fluid dynamics, the N-dimensional isentropic compressible Euler equations are expressed as follows:

$$\begin{cases} \rho_t + \nabla \cdot (\rho \vec{u}) = 0, \\ \rho[\vec{u}_t + (\vec{u} \cdot \nabla)\vec{u}] + K\nabla\rho^\gamma = 0, \end{cases} \tag{1}$$

where $\rho = \rho(t, \vec{x})$ denotes the density of the fluid, $\vec{u} = \vec{u}(t, \vec{x}) = (u_1, u_2, \ldots, u_N) \in R^N$ is the velocity, $\vec{x} = (x_1, x_2, \ldots, x_N) \in R^N$, that we use $x_1 = x$, $x_2 = y$ and $x_3 = z$ for $N \leq 3$ and $K > 0$, $\gamma \geqslant 1$ are constants.

Basically, these Euler equations are a set of equations that govern the inviscid flow of a fluid. The first and second equations of (1) represent, respectively, the conservation of mass and the momentum of the fluid.

The Euler equations have applications in many mathematical physics subjects, such as fluids, plasmas, condensed matter, astrophysics, oceanography and atmospheric dynamics. For real-life applications, they can be used in the study of turbulence, weather forecasting and the prediction of earthquakes and the explosion of supernovas.

The Euler equations are the basic model of shallow water flows [7]. In [9], they are used to model the super-fluids produced by Bose–Einstein condensates in the dilute gases of alkali metals, in which identical gases do not interact at very low temperatures. However, at the microscopic level, fluids or gases are formed by many tiny discrete molecules or particles that collide with one another. As the cost of directly calculating the particle-to-particle or molecule-to-molecule evolution of the fluids on a large scale is expensive, approximation methods are needed to considerably simplify the process. An example of an approximation method is given in [5], where the Euler equations are used to describe the behavior of fluids at the

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statistical limit of a large number of small ideal molecules or particles by ignoring the less influential effects, such as self-gravitational forces and the relativistic effect. The detailed derivation of the Euler equations can be found in [11,6].

The construction of analytical or exact solutions is an important area in mathematical physics and applied mathematics, as it can further classify nonlinear phenomena. For non-rotational flows, Makino first obtained the radial symmetry solutions for the Euler equation (1) in R^N in 1993 [12]. A number of special solutions for these equations [10,16] were subsequently obtained. Yuen later obtained a class of self-similar solutions with elliptical symmetry in 2012 [17]. For rotational flows, Zhang and Zheng constructed explicitly rotational solutions for the Euler equations with $\gamma = 2$ and $N = 2$ in 1997 [19]. In 2014, Yuen obtained a class of rotational solutions for the compressible Euler equation (1) for $\gamma > 1$ in 2D in [18]:

$$\begin{cases} \rho = \dfrac{\max\left(\left(-\frac{\lambda(\gamma-1)}{2K\gamma}\eta+\alpha\right)^{\frac{1}{\gamma-1}},0\right)}{a^2(t)}, \\ u_1 = \frac{\dot{a}(t)}{a(t)}x - \frac{\xi}{a^2(t)}y, \\ u_2 = \frac{\xi}{a^2(t)}x + \frac{\dot{a}(t)}{a(t)}y, \\ \ddot{a}(t) - \frac{\xi^2}{a^3(t)} = \frac{\lambda}{a^{2\gamma-1}(t)}, a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \end{cases} \tag{2}$$

with a self-similar variable $\eta = \frac{x^2+y^2}{a^2(t)}$ and arbitrary constants $\lambda, \alpha \geqslant 0, \xi \neq 0, a_0$ and a_1.

For the physical applications of the similar solutions for the compressible Euler equations, readers may refer to [13,2–4,8].

Based on the works in [17,18], we obtain novel rotational and self-similar solutions for the 3D compressible Euler equation (1).

Theorem 1. *For the compressible Euler equation (1) in R^3, there exists a family of rotational and self-similar solutions*

$$\begin{cases} \rho = \frac{f(s)}{a^2(t)b(t)}, \\ u_1 = \frac{\dot{a}(t)}{a(t)}x - \frac{\xi}{a^2(t)}y, \\ u_2 = \frac{\xi}{a^2(t)}x + \frac{\dot{a}(t)}{a(t)}y, \\ u_3 = \frac{\dot{b}(t)}{b(t)}z, \end{cases} \tag{3}$$

with a variable $s = \frac{x^2+y^2}{a^2(t)} + \frac{z^2}{b^2(t)}$ *and*

$$f(s) = \begin{cases} \alpha e^{-\frac{\lambda}{2K}s} & for \quad \gamma = 1, \\ \max\left(\left(-\frac{\lambda(\gamma-1)}{2K\gamma}s+\alpha\right)^{\frac{1}{\gamma-1}},0\right) & for \quad \gamma > 1, \end{cases} \tag{4}$$

and the corresponding Emden system

$$\begin{cases} \ddot{a}(t) - \frac{\xi^2}{a^3(t)} = \frac{\lambda}{a^{2\gamma-1}(t)b^{\gamma-1}(t)}, a(0) = a_0 > 0, \quad \dot{a}(0) = a_1, \\ \ddot{b}(t) = \frac{\lambda}{a^{2\gamma-2}(t)b^{\gamma}(t)}, \quad b(0) = b_0 > 0, \quad \dot{b}(0) = b_1, \end{cases} \tag{5}$$

where $\xi \neq 0, \lambda, \alpha \geqslant 0, a_0, a_1, b_0$ and b_1 are arbitrary constants. In particular, if any one following condition is further fulfilled,

(1) *with* $\gamma = 1$;
(2) *with* $\gamma > 1$,
(2a) $\lambda \leq 0$ *or*
(2b) $\lambda > 0$ *and* $\gamma < 2$,

solutions (3)–(5) are C^1.

Remark 2. Solutions (3)–(5) of the compressible Euler equation (1) in R^3 are very efficient for testing the accuracy of many numerical solutions about vortices in computational fluid dynamics. In particular, the 3D rotational solutions provide concrete reference examples for modeling typhoons in oceans.

Remark 3. For the compressible Euler equations (1) in R^3, the rotational solutions (3)–(5) correspond to Yuen's irrotational and elliptic solutions in R^3 [17] as well as rotational and radial solutions in R^2 [18].

2. Rotational and self-similar solutions

To prove Theorem 1, we need the following novel lemma for the three-dimensional mass equation $(1)_1$.