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# Analytic studies on a generalized inhomogeneous higher-order nonlinear Schrödinger equation for the Heisenberg ferromagnetic spin chain

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For the dynamics of spins in an inhomogeneous classical continuum biquadratic Heisenberg ferromagnetic spin chain with the deformation of the inhomogeneous Heisenberg ferromagnetic spin system through a space curve formalism, we work on the behavior of solitons described by a generalized inhomogeneous higher-order nonlinear Schrödinger equation. Upon the introduction of an auxiliary function, bilinear forms, analytic one- and two-soliton solutions are derived via the Hirota method. We find that the inhomogeneous parameters can affect the amplitude of the soliton, and also see the existence of explode–decay soliton. Asymptotic analysis is carried out on the two-soliton solutions. Effects of the linear inhomogeneities on the one and two solitons are investigated graphically and analytically. Soliton amplitude and peak position are related to the inhomogeneous coefficients of the equation. Interaction between two solitons follows the attraction–repulsion process.



## 1. Introduction

The nonlinear Schrödinger (NLS)-type equations can be used to describe the nonlinear phenomena in optics, plasmas physics, hydrodynamics and condensed matter physics [1–9]. The NLS-type solutions, in the inhomogeneous media, have their potential applications in describing, e.g., the ultrashort optical pulses in long-distance communication [10–12] and spin dynamics in an inhomogeneous classical continuum biquadratic Heisenberg ferromagnetic spin chain [13,14]. For instance, to describe the effect of twist inhomogeneity on the soliton spin excitations in a one-dimensional inhomogeneous helimagnet in the semiclassical limit, a generalized perturbed fourth-order NLS equation has been obtained [15]. To investigate the dynamics of matter-wave solitons in the presence of a spatially-varying atomic scattering length and nonlinearity, bright and dark solitons of a generalized NLS equation have been studied [16]. Integrability of the classical one-dimensional Heisenberg inhomogeneous ferromagnetic spin chain and effect of the inhomogeneity on the soliton in a higher-order NLS equation have been analyzed [17].

In this paper, we will investigate an fourth-order NLS equation with linearly $x$-dependent coefficients [18–28],

$$iq_t + \varepsilon q_{xxxx} + 8\varepsilon|q|^2 q_{xx} + 2\varepsilon q^2 q^*_{xx} + 4\varepsilon|q_x|^2 q + 6\varepsilon q^* q_x^2 + 6\varepsilon|q|^4 q + (\eta q)_{xx} + 2q\left(\eta|q|^2 + \int_{-\infty}^{x} \eta_x |q|^2 dx'\right) - i(\rho q)_x = 0, \qquad (1)$$

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with $\eta = a_1 x + b_1, \quad \rho = a_2 x + b_2$, which can be associated with the motion of a nonlinear string of constant length or a space curve with an equation of motion in the Euclidean space [18,20,23], where $q(x,t)$ is the energy density of the spin system through the curvature and torsion of the space curve, $t$ and $x$ are the normalized time and propagation distance, respectively, $\eta$ and $\rho$ represent the inhomogeneities in the medium and are the linear functions of $x$ [18], $a_1, a_2, b_1$ and $b_2$ are the real parameters, $\varepsilon = d^2/12$, with $d$ being the lattice distance, is related to the effect of discreteness and higher-order magnetic interactions [19], the asterisk and subscripts denote the complex conjugate and partial derivatives, respectively. Eq. (1) with constant coefficients can describe the inhomogeneities or nonuniformities of materials in optical fibers, plasma for laser radiations and condensed matter physics [18–20]. Integrable deformation of the inhomogeneous Heisenberg ferromagnet model has been investigated via the prolongation structure and Eq. (1), as the corresponding equivalent equation, has been obtained [20]. $\varepsilon$ has been claimed to affect the soliton velocity: when the wavenumber of the soliton is given, the soliton velocity changes linearly with $\varepsilon$ [21,22]. Bright soliton solutions for Eq. (1) have been obtained from the associated linear eigenvalue problems via the gauge transformation [18]. Eq. (1) has been proven to be integrable, possessing the Lax pair and Bäcklund transformation [23]. Magneto-optical effects on a biquadratic ferromagnet and dynamics of Eq. (1) have been investigated [24]. Nonlinear spin dynamics of a one-dimensional anisotropic continuum Heisenberg ferromagnetic spin chain with the octupole–dipole interaction in the semiclassical continuum limit in Eq. (1) has been presented [25]. Stability of the embedded solitons in Eq. (1) has been studied, and embedded solitons under internal perturbations have been derived through the conserved quantities [26]. Due to the different choices of $\eta$ and $\rho$, Eq. (1) can be reduced to different kinds of the NLS-type equations [27,28]: When $\rho = 0$, effect of the nonlinear inhomogeneity in bilinear and biquadratic exchange interactions on the spin soliton in a classical continuum Heisenberg ferromagnetic spin chain has been studied via a perturbation analysis [27]; When $\eta = \rho = 0$, Eq. (1) describes the way that the non-Kerr nonlinearity influences the soliton propagation [28]; Eq. (1) with $\rho = 0$ has been studied analytically with the discovery that the linear coefficients $a_1$ and $b_1$ are related to the soliton amplitude and position, respectively [19].

However, the case with the inhomogeneous $\rho$ has not been considered as yet. With the help of the auxiliary function, we will try to obtain the one- and two-soliton solutions and analyze the evolution and interaction of solitons. In Section 2, with the Hirota method [29–31], we will obtain the bilinear forms for Eq. (1). Through symbolic computation [32–35], one- and two-soliton solutions for Eq. (1) will be derived. In Section 3, we will investigate certain dynamics of solitons analytically and graphically. Choosing different values of $\eta$ and $\rho$ for Eq. (1), we will show the effects of linear inhomogeneity parameters on the evolution of one soliton and interaction of solitons. Section 4 will be our conclusions.

## 2. Bilinear forms and soliton solutions

Substituting the dependent variable transformation

$$q = \frac{g(x,t)}{f(x,t)} \tag{2}$$

into Eq. (1), we will obtain

$$\begin{aligned} & i\frac{D_t g\cdot f}{f^2} + \varepsilon\left[\frac{D_x^4 g\cdot f}{f^2} - \frac{gD_x^4 f\cdot f}{f^3} + 6\frac{g}{f}\left(\frac{D_x^2 f\cdot f}{f^2}\right)^2 - 6\frac{D_x^2 g\cdot fD_x^2 f\cdot f}{f^4}\right] - ia_2\frac{g}{f} + 2\varepsilon\frac{g^2}{f^2}\left(\frac{D_x^2 g^*\cdot f}{f^2} - \frac{g^* D_x^2 f\cdot f}{f^3}\right) + 4\varepsilon\frac{gD_x g\cdot fD_x g^*\cdot f}{f^5} \\ & + 6\varepsilon\frac{g^*}{f}\left(\frac{D_x g\cdot f}{f^2}\right)^2 + 6\varepsilon\frac{g^3 g^{*2}}{f^5} + 2a_1\frac{D_x g\cdot f}{f^2} + (a_1 x + b_1)\left(\frac{D_x^2 g\cdot f}{f^2} - \frac{gD_x^2 f\cdot f}{f^3}\right) + 2(a_1 x + b_1)\frac{g^2 g^*}{f^3} + 2a_1\frac{gf_x}{f^2} \\ & + 8\varepsilon\frac{gg^*}{f^2}\left(\frac{D_x^2 g\cdot f}{f^2} - \frac{gD_x^2 f\cdot f}{f^3}\right) - i(a_2 x + b_2)\frac{D_x g\cdot f}{f^2} = 0, \end{aligned} \tag{3}$$

where $g(x,t)$ is a complex differentiable function, $f(x,t)$ is a real one, and $D_x$ and $D_t$ are the bilinear derivative operators defined by [31]

$$D_x^m D_t^n (a\cdot b) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,t) b(x',t')|_{x'=x, t'=t}, \tag{4}$$

with $x'$ and $t'$ being the formal variables, $a(x,t)$ as the function of $x$ and $t$, $b(x',t')$ as the function of $x'$ and $t'$, $m = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$.

With $\frac{D_x^4 f\cdot f}{f^2} = (D_x^2 f\cdot f/f^2)_{xx} + 3(D_x^2 f\cdot f/f^2)^2$ and $D_x^2 f\cdot f = 2|g|^2$ [19], Eq. (3) can be rewritten in the following bilinear forms:

$$\begin{aligned} & \left[iD_t + 2a_1 D_x + (a_1 x + b_1)D_x^2 + \varepsilon D_x^4 - iu_2 - i(a_2 x + b_2)D_x\right](g\cdot f) + 2a_1 g f_x = 3\varepsilon s g^*, \\ & D_x^2 g\cdot g = sf, \\ & D_x^2 f\cdot f = 2|g|^2, \end{aligned} \tag{5}$$

where $s(x,t)$ is the complex auxiliary function. Based on Eq. (5), we can derive the soliton solutions via the following power series expansions for $g, f$ and $s$,