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On cusped solitary waves in finite water depth

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ABSTRACT

It is well-known that the Camassa–Holm (CH) equation admits both of the peaked and cusped solitary waves in shallow water. However, it was an open question whether or not the exact wave equations can admit them in finite water depth. Besides, it was traditionally believed that cusped solitary waves, whose 1st-derivative tends to infinity at crest, are essentially different from peaked solitary ones with finite 1st-derivative. Currently, based on the symmetry and the exact water wave equations, Liao [1] proposed a unified wave model (UWM) for progressive gravity waves in finite water depth. The UWM admits not only all traditional smooth progressive waves but also the peaked solitary waves in finite water depth: in other words, the peaked solitary progressive waves are consistent with the traditional smooth ones. In this paper, in the frame of the linearized UWM, we give, for the first time, some explicit expressions of cusped solitary waves in finite water depth, and besides reveal a close relationship between the cusped and peaked solitary waves: a cusped solitary wave is consist of an infinite number of peaked solitary ones with the same phase speed, so that it can be regarded as a special peaked solitary wave. This also well explains why and how a cuspon has an infinite 1st-derivative at crest. Besides, it is found that, when wave height is small enough, the effect of nonlinearity is negligible for the interaction of peaked waves so that these explicit expressions are good enough approximations of peaked/cusped solitary waves in finite water depth. In addition, like peaked solitary waves, the vertical velocity of a cusped solitary wave in finite water depth is also discontinuous at crest ($x = 0$), and especially its phase speed has nothing to do with wave height, too. All of these would deepen and enrich our understandings about the cusped solitary waves.

1. Introduction

The smooth solitary surface wave was first reported by John Scott Russell [2] in 1844. Since then, various types of solitary waves have been found. The mainstream models of shallow water waves, such as the Boussinesq equation [3], the KdV equation [4], the BBM equation [5] and so on [6,7], admit dispersive *smooth* periodic/solitary progressive waves with permanent form: the wave elevation is *infinitely* differentiable *everywhere*. Especially, the phase speed of the smooth waves is highly dependent upon wave height: the larger the wave height of a smooth progressive wave, the faster it propagates. The only

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exception is the limiting wave with the highest amplitude, which has a sharp crest, as pointed out by Stokes. However, Stokes limiting wave has never been observed in practice. Nowadays, the smooth amplitude-dispersive periodic/solitary waves are the mainstream of researches in water waves.

In 1993, Camassa and Holm [8] proposed the celebrated Camassa–Holm (CH) equation for shallow water waves, and first reported the so-called peaked solitary wave, called peakon, which has a peaked crest with a discontinuous (but finite) 1st-order derivative at crest. This is a breakthrough in water wave theories, since it opens a new field of research in the past 20 years. Physically, different from the KdV equation and Boussinesq equation, the CH equation can model phenomena of not only soliton interaction but also wave breaking [9]. Mathematically, the CH equation is integrable and bi-Hamiltonian, therefore possesses an infinite number of conservation laws in involution [8]. Besides, it is associated with the geodesic flow on the infinite dimensional Hilbert manifold of diffeomorphisms of line [9]. Thus, the CH equation has lots of intriguing physical and mathematical properties. It is even believed that the CH equation "has the potential to become the new master equation for shallow water wave theory" [10]. In addition, Kraenkel and Zenchuk [11] reported the cusped solitary waves of the CH equation, called cuspon. The so-called cuspon is a kind of solitary wave with the 1st derivative going to *infinity* at crest. Note that, unlike a peakon that has a *finite* 1st derivative, a cuspon has an *infinite* 1st derivative at crest. Thus, it was traditionally believed that peakons and cuspons are completely *different* two kinds of solitary waves.

However, the CH equation is a simplified model of water waves in *shallow* water. It was an open question whether or not the exact wave equations admit the peaked and cusped solitary waves in *finite* water depth. For example, the velocity distribution of peaked/cusped solitary waves in the vertical direction was unknown, since it can not be determined by a wave model in shallow water (such as the CH equation). Currently, based on the symmetry and the exact wave equations, Liao proposed a unified wave model (UWM) for progressive gravity waves in finite water depth with permanent form [1]. It was found that the UWM admits not only all traditional smooth periodic/solitary waves but also the peaked solitary waves in finite water depth, even including the famous peaked solitary waves of the CH equation as its special case. Therefore, the UWM unifies both of the smooth and peaked solitary waves in finite water depth, for the first time. In other words, the progressive peaked solitary waves in finite water depth are consistent with the traditional smooth waves, and thus are as acceptable and reasonable as the smooth ones.

In this article, we first give an closed-form expression of cusped solitary waves in finite water depth by means of the linearized UWM. Then, we show that, when wave height is small, the effect of nonlinearity is small and only near the crest so that it is negligible. Thus, a cusped solitary wave might be regarded as an infinite number of peaked solitary ones. This reveals, for the first time to the best of my knowledge, a simple but elegant relationship between the peaked and cusped solitary waves in finite water depth.

2. Cusped solitary waves in finite water depth

Let us first describe "the unified wave model" (UWM) [1] briefly. Consider a progressive gravity wave propagating on a horizontal bottom in a *finite* water depth D, with a constant phase speed c and a permanent form. For simplicity, the problem is solved in the frame moving with the phase speed c. Let x, z denote the horizontal and vertical dimensionless co-ordinates (using the water depth D as the characteristic length), with $x = 0$ corresponding to the wave crest, $z = -1$ to the bottom, and the z axis upward, respectively. Assume that the wave elevation $\eta(x)$ has a symmetry about the crest, the fluid in the interval $x > 0$ (and $x < 0$) is inviscid and incompressible, the flow in $x > 0$ (and $x < 0$) is irrotational, and surface tension is neglected. Here, it should be emphasized that, different from all traditional wave models, the flow at $x = 0$ is *not* absolutely necessary to be irrotational. Let $\phi(x,z)$ denote the velocity potential. All of them are dimensionless using D and $\sqrt{gD}$ as the characteristic scales of length and velocity, where g is the acceleration due to gravity. In the frame of the UWM, the velocity potential $\phi(x,z)$ and the wave elevation $\eta(x)$ are first determined by the exact wave equations (i.e. the Laplace equation $\nabla^2\phi = 0$, the two nonlinear boundary conditions on the unknown free surface η, the bed condition and so on) only in the interval $x \in (0, +\infty)$, and then extended to the whole interval $(-\infty, +\infty)$ by means of the symmetry

$$\eta(-x) = \eta(x), \quad u(-x,z) = u(x,z), \quad v(-x,z) = -v(x,z),$$

which enforces the additional restriction condition $v(0,z) = 0$. It should be emphasized that, in the frame of the UWM, the flow at $x = 0$ is *not* necessarily irrotational, so that the UWM is more general: this is the reason why the UWM can admit both of the smooth and peaked solitary waves. For details, please refer to Liao [1].

When wave height is small enough, the effect of nonlinearity is small and besides only near the crest, so that the nonlinearity is negligible, as shown in Section 3. Therefore, we first consider the linearized UWM in this section.

In the interval $(0, +\infty)$, the governing equation $\nabla^2\phi(x,z) = 0$ with the bed condition $\phi_z(x,-1) = 0$ has two kinds of general solutions [12], where the subscript denotes the differentiation with respect to z. One is

$$\cosh[nk(1+z)]\sin(nkx),$$

corresponding to the smooth periodic waves with the dispersive relation

$$\alpha^2 = \frac{\tanh(k)}{k} \leqslant 1, \tag{1}$$

where $\alpha = c/\sqrt{gD}$ is the dimensionless phase speed, k is wave number and n is an integer, respectively. The other is