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Unique solvability of a steady-state complex heat transfer model

Andrey E. Kovtanyuk [a,b,*], Alexander Yu. Chebotarev [a,b], Nikolai D. Botkin [c], Karl-Heinz Hoffmann [c]

[a] Far Eastern Federal University, Sukhanova st. 8, 690950 Vladivostok, Russia
[b] Institute for Applied Mathematics FEB RAS, Radio st. 7, 690041 Vladivostok, Russia
[c] Technische Universität München, Zentrum Mathematik, Boltzmannstr. 3, D-85747 Garching bei München, Germany

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ABSTRACT

The problem of radiative–conductive–convective heat transfer in a three-dimensional domain is studied in the framework of the diffusion (P_1) steady-state approximation. The unconditional unique solvability of this nonlinear model is proved in the case of Robin-type boundary conditions for the temperature and the mean intensity function. An iterative algorithm for the numerical solution of the model is proposed. Numerical examples demonstrating the importance of the radiative heat transfer at high temperatures are presented.

1. Introduction

The interest in studying problems of complex heat transfer (where the radiative, convective, and conductive contributions are simultaneously taken into account) is motivated by their importance for many engineering applications. Here, the following examples can be mentioned: modeling the heat transfer in combustion chambers and industrial furnaces, estimating the efficiency of cooling systems, predicting heat transfer in glass manufacturing, control of thermal processes in optical fiber production, etc.

The common feature of such processes is the radiative heat transfer dominating at high temperatures. The radiative heat transfer equation (RTE) is a first order integro-differential equation governing the radiation intensity. The radiation traveling along a path is attenuated as a result of absorption and scattering, and it is amplified due to emission and incoming scattering along the path. The precise derivation an analysis of such models can be found in the monograph [1].

Solutions to the RTE can be represented in the form of the Neumann series whose terms are powers of an integral operator applied to a certain start function. The terms can be calculated using a Monte Carlo method, which may be interpreted as tracking the history of energy bundles from emission to adsorption at a surface or within a participating medium. The method assumes that the bundles start from random points, propagate in random directions, and show the energy exchange due to random scattering (see e.g. [2,3]).

* Corresponding author at: Far Eastern Federal University, Sukhanova st. 8, 690950 Vladivostok, Russia.
E-mail addresses: kovtanyuk.ae@dvfu.ru (A.E. Kovtanyuk), cheb@iam.dvo.ru (A.Yu. Chebotarev), botkin@ma.tum.de (N.D. Botkin), hoffmann@ma.tum.de (K.-H. Hoffmann).

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Examples of numerical analysis of one-dimensional heat transfer models coupled with the RTE can be found in [3–5]. In [6] the unconditional unique solvability of one-dimensional steady-state complex heat transfer problems is proved. Papers [7,8] state conditional unique solvability in three dimensions.

A way of avoiding the solution of the integro-differential RTE is the use of expansions of the local intensity in terms of spherical harmonics, with truncation to N terms in the series, and substitution into the moments of the differential form of the equation of transfer (see e.g. [1]). This approach leads to the P_N approximations, where N is the order of the approximations. As N approaches infinity the solutions obtained became exact. Usually, the odd orders are employed, especially P_1 and P_3. Going to P_5, increase the accuracy, but the complexity of the calculations becomes unacceptable. It is important that the P_N approximation of the RTE is a local partial differential equation supplied with appropriate boundary and initial conditions. The work [9] studies the diffuse (P_1) approximations of three-dimensional complex heat transfer models. Asymptotical expansions are proposed, and computations related to the glass manufacturing process are performed. Paper [10] considers a three-dimensional transient SP_N (some modification of P_N) approximate model and presents extensive numerical simulations. In [11], a three-dimensional transient SP_1 model is considered, theoretical backgrounds for the development of optimal control techniques are established, and numerical simulations are conducted.

Especially interesting is the use of steady state P_N and, in particular, P_1 approximations because they describe steady-state temperature distributions and do not require high computational efforts. The main difficulty related to steady-state models is the absence of theoretical results on the unique solvability in three dimensions. In the one-dimensional case, the conditional unique solvability is proved in [12]. For three-dimensional models, the existence and conditional uniqueness of solutions is proved in [13–15]. Here, sufficient conditions of the uniqueness are a small size of the heat-transfer domain and a large convection velocity. The last paper also addresses a control problem and formulates necessary optimality conditions.

In the present paper, new results on the uniqueness are obtained. Namely, the unconditional unique solvability, without smallness or largeness assumptions, is proved in the case of Robin-type boundary conditions for the temperature and the mean intensity function.

2. Model description

The following steady-state normalized diffusion (P_1) model (see [1]) describing radiative, conductive, and convective heat transfer in a bounded domain $G \subset \mathbb{R}^3$ is under consideration:

$$-a\Delta\theta + \mathbf{v}\cdot\nabla\theta + b\kappa_a|\theta|\theta^3 = b\kappa_a\varphi, \quad -\alpha\Delta\varphi + \kappa_a\varphi = \kappa_a|\theta|\theta^3. \tag{1}$$

Here, θ is the normalized temperature, φ the normalized radiation intensity averaged over all directions, $\mathbf{v}$ a given divergence free velocity field, and κ_a the absorption coefficient. The constants a, b, and α are given by the formulas

$$a = \frac{k}{\rho c_v}, \quad b = \frac{4\sigma n^2 T_{max}^3}{\rho c_v}, \quad \alpha = \frac{1}{3\kappa - A\kappa_s},$$

where k is the thermal conductivity, c_v the specific heat capacity, ρ the density, σ the Stefan–Boltzmann constant, n the refractive index, T_{max} the maximum temperature in the unnormalized model, $\kappa := \kappa_s + \kappa_a$ the extinction coefficient (total attenuation factor), and κ_s the scattering coefficient. The coefficient $A \in [-1, 1]$ describes the anisotropy of scattering.

The following boundary conditions on $\Gamma := \partial G$ are assumed:

$$a\partial_n\theta + \gamma(\theta - \Theta_0)|_\Gamma = 0, \quad \alpha\partial_n\varphi + \beta(\varphi - \Theta_0^4)|_\Gamma = 0, \tag{2}$$

where Θ_0, β, and γ are given functions defined on Γ, and the symbol ∂_n denotes the derivative in the outward normal direction.

3. A priori estimates of weak solutions

Assume that G is a Lipschitz bounded domain with the boundary Γ consisting of a finite set of smooth pieces. Let $(\cdot,\cdot)$ and $\|\cdot\|$ be the inner product and the norm of the space $L^2(\Omega)$, respectively. The Lebesgue space $L^p(\Omega), 1 \leqslant p \leqslant \infty$, and the Sobolev space $H^m(\Omega) := W_2^m(\Omega)$, $m \in \mathbb{N}$, are supposed to be defined in the conventional way.

Suppose that the model data satisfy the following conditions:

(i) $\mathbf{v} \in H^1(G) \cap L^\infty(G), \quad \nabla\cdot\mathbf{v} = 0$;
(ii) $\Theta_0, \beta, \gamma \in L^\infty(\Gamma),\ 0 \leqslant \Theta_0 \leqslant M,\ \ \beta \geqslant \beta_0 > 0,\ \gamma \geqslant \gamma_0 > 0$;
(iii) $\gamma + (\mathbf{v}\cdot\mathbf{n}) \geqslant c_0 > 0$ on the boundary part where $(\mathbf{v}\cdot\mathbf{n}) < 0$.

Here, M, β_0, γ_0, and c_0 are positive constants.

Definition 1. A pair $(\theta, \varphi) \in H^1(G) \times H^1(G)$ is called weak solution of the problem (1) and (2) if