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On constructing accurate approximations of first integrals for difference equations

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ABSTRACT

In this paper, a perturbation method based on invariance factors and multiple scales will be presented for weakly nonlinear, regularly perturbed systems of ordinary difference equations. Asymptotic approximations of first integrals will be constructed on long iteration-scales, that is, on iteration-scales of order ε^{-1}, where ε is a small parameter. It will be shown that all invariance factors have to satisfy a functional equation. To show how this perturbation method works, the method is applied to a Van der Pol equation, and a Rayleigh equation. It will be explicitly shown for the first time in the literature how these multiple scales should be introduced for systems of difference equations to obtain very accurate approximations of first integrals on long iteration-scales.

1. Introduction

Nowadays, the multiple time-scales perturbation method for differential equations is a well developed, well accepted, and a very popular method to approximate solutions of weakly nonlinear differential equations. This method was developed in the period 1935–1962 by Krylov and Bogoliubov, Kuzmak, Kevorkian, and Cole, Cochran, and Mahony. In the early 1970s, Nayfeh popularized this method by writing many papers and books on this subject (see for instance [1]). More recent books on this method and its historical development are, for instance, the books by Andrianov and Manevitch [2], Holmes [3], Kevorkian and Cole [4], Murdock [5], and Verhulst [6].

For scientists and engineers the analysis of nonlinear dynamical systems is an important field of research since the solutions of these systems can exhibit counterintuitive and sometimes unexpected behavior. To obtain useful information from these systems the construction of (approximations of) first integrals by means of computing (approximate) integrating factors can play an important role. In [7] a perturbation method based on integrating vectors and multiple scales was presented for regularly perturbed differential equations.

Recently, first integrals, invariants and Lie group theory for ordinary difference equations (OΔEs) obtained a lot of attention in the literature (see for instance the list of references in [8]). Van Horssen [9] developed a perturbation method for a single first order difference equation based on invariance factors. Also recently, the fundamental concept of invariance factors for OΔEs to obtain invariants (or first integrals) for OΔEs has been presented in [8]. In [10] a perturbation method based on invariance factors has been presented for regularly perturbed systems of ordinary difference equations (OΔEs). Using straightforward (but naive) approximations for the invariance vectors it has been shown how approximations of first integrals (including error-estimates) can be obtained. It turned out that some of the approximations for the first integrals are valid

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on the iteration scales for which the original problem for the system of OΔEs has been proved to be well-posed. However, some of the approximations are only valid on a much smaller iteration scale. This is caused by the use of straightforward, naive approximations for the invariance factors, and as a well-known consequence in perturbation theory, the so-called secular terms occur in the approximations. To avoid secular terms in the approximations for the solution of a difference equation, the well-known perturbation method based on multiple (iteration-) scales can sometimes be used successfully. The development of the multiple scales perturbation method for ordinary difference equations (OΔEs) started in 1960 with the work of Torng [11]. From the results obtained in [11], it is clear that the solution of a weakly perturbed (non) linear OΔEs behaves differently on different iteration scales. In 1977, Hoppensteadt and Miranker introduced in [12] the multiple scales perturbation method for OΔEs. It is interesting to notice that for difference equations a formulation of the multiple-scales methods completely in terms of difference operators has also recently been developed in [13]. To apply this method, an additional, new (iteration-) scale is usually introduced in the approximation for the solution. For instance $\tau_n = \varepsilon n$, where n is the original fast iteration-scale, and where τ_n is the new slow iteration-scale. In the multiple (iteration-) scales perturbation method these variables n and τ_n will be treated as independent variables. Now it should be observed that $\tau_{n+1} = \tau_n + \varepsilon$, and so by introducing a new (iteration-) scale, one adds in fact a new difference equation to the original system of OΔEs (note that τ_n will now be treated as a variable dependent on n). This simple observation led in [13] to a perturbation method based on invariance vectors and multiple scales. It turned out that this perturbation method is consistent and straightforward. Moreover, it turned out that secular terms can be avoided and that error-estimates can easily be obtained on long iteration-scales.

It has been shown in [8] that in finding invariants for a system of first order difference equations all invariance factors have to satisfy a functional equation (for more information on functional equations and how to solve some of them we refer the reader to [14–19]). The aim of this paper is to construct asymptotic approximations of first integrals for a system of first order OΔEs. After presenting the concepts, we will explicitly show how highly accurate approximations of invariants for a second order, weakly nonlinear difference equation (with a Van der Pol type of nonlinearity), and also for a weakly nonlinear Rayleigh equation can be constructed by using the multiple scales perturbation method. Finally, it should be remarked that the advantage of the method of invariance factors is that the method can be applied to problems which cannot be integrated or be solved completely when the small parameter is zero. For such perturbed, essentially nonlinear problems one can find approximations of some of the first integrals by applying the perturbation method based on invariance factors. Classical perturbation methods can usually only be applied to problems which are linear when the small parameter is zero. So, the perturbation method as presented in this paper can be applied to a much larger class of problems as compared to the classical perturbation methods.

The outline of this paper is as follows. In Section 2 the concept of invariance factors for a system of first order OΔEs will be given, and in Section 3 some of the first results in the development of a multiple scales perturbation method for a system of OΔEs based on invariance factors will be presented. In Sections 4 and 5 approximations of first integrals for systems of weakly nonlinear OΔEs of Van der Pol type, and of Rayleigh type, respectively, will be constructed. Finally, in Section 6 of this paper some conclusions will be drawn and some future directions for research will be indicated.

2. On invariance factors for OΔEs

In this section we are going to present an overview of the concept of invariance factors (or vectors) for a system of k first order OΔEs (where k is fixed and $k \in \mathbb{N}$), for a reference concerning this concept, we refer the reader to [9]. Consider

$$\underline{x}_{n+1} = \underline{f}(\underline{x}_n, n), \tag{1}$$

for $n = 0, 1, 2, \ldots$, and where $\underline{x}_n = (x_{1,n}, x_{2,n}, \ldots, x_{k,n})^T$, $\underline{f} = (f_1, f_2, \ldots, f_k)^T$ in which the superscript indicates the transpose, and where $f_i = f_i(\underline{x}_n, n) = f_i(x_{1,n}, x_{2,n}, \ldots, x_{k,n}, n)$ are sufficiently smooth functions (for $i = 1, 2, \ldots, k$). We also assume that an invariant for (1) can be represented by

$$I(\underline{x}_{n+1}, n+1) = I(\underline{x}_n, n) = \text{constant} \Longleftrightarrow \Delta I(\underline{x}_n, n) = 0. \tag{2}$$

We now try to find an invariant for (1). By multiplying each ith equation, in (1), with a factor $\mu_i(\underline{x}_{n+1}, n+1) = \mu_i(\underline{f}(\underline{x}_n, n), n+1)$ for $i = 1, 2, \ldots, k$ and by adding the resulting equations we obtain

$$\underline{\mu}(\underline{x}_{n+1}, n+1) \cdot \underline{x}_{n+1} = \underline{\mu}(\underline{f}(\underline{x}_n, n), n+1) \cdot \underline{f}(\underline{x}_n, n), \tag{3}$$

where $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)^T$, and $\mu_i = \mu_i(\underline{x}_n, n) = \mu_i(x_{1,n}, x_{2,n}, \ldots, x_{k,n}, n)$ for $i = 1, 2, \ldots, k$. In fact, when $\underline{\mu}$ is an invariance vector we now obtain an exact difference equation (2). The relationship between I and $\underline{\mu}$ follows from the equivalence of (2) and (3), yielding

$$\begin{cases} I(\underline{x}_{n+1}, n+1) = \underline{\mu}(\underline{x}_{n+1}, n+1) \cdot \underline{x}_{n+1}, \\ I(\underline{x}_n, n) = \underline{\mu}(\underline{f}(\underline{x}_n, n), n+1) \cdot \underline{f}(\underline{x}_n, n). \end{cases} \tag{4}$$

By reducing the index $n+1$ by 1 in the first part of (4) I can be eliminated from (4), and then it follows that all invariance vectors for the system of difference equations (1) have to satisfy the functional equation

$$\underline{\mu}(\underline{x}_n, n) \cdot \underline{x}_n = \underline{\mu}(\underline{f}(\underline{x}_n, n), n+1) \cdot \underline{f}(\underline{x}_n, n). \tag{5}$$