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A sum operator method for the existence and uniqueness of positive solutions to Riemann–Liouville fractional differential equation boundary value problems ☆

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A B S T R A C T

In this paper, we are concerned with the existence and uniqueness of positive solutions for the following fractional boundary value problems given by

$$-D_{0+}^{\alpha}u(t) = f(t,u(t)) + g(t,u(t)), \quad 0 < t < 1, \ 3 < \alpha \leqslant 4,$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, subject either to the boundary conditions $u(0) = u'(0) = u''(0) = u''(1) = 0$ or $u(0) = u'(0) = u''(0) = 0$, $u''(1) = \beta u''(\eta)$ for $\eta, \beta\eta^{\alpha-3} \in (0,1)$. Our analysis relies on a fixed point theorem of a sum operator. Our results can not only guarantee the existence of a unique positive solution, but also be applied to construct an iterative scheme for approximating it. Two examples are given to illustrate the main results.

1. Introduction

Fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics, engineering and biological sciences, etc.; see [1–13] for example. In recent years, the study of positive solutions for fractional differential equation boundary value problems has attracted considerable attention, and fruits from research into it emerge continuously. For a small sample of such work, we refer the reader to [14–22] and the references therein. On the other hand, the uniqueness of positive solutions for nonlinear fractional differential equation boundary value problems has been studied by some authors, see [17–19,21,23] for example.

In this paper, we consider a nonlinear differential equation of fractional order having the form

$$-D_{0+}^{\alpha}u(t) = f(t,u(t)) + g(t,u(t)), \quad 0 < t < 1, \ 3 < \alpha \leqslant 4, \tag{1.1}$$

where D_{0+}^{α} is the standard Riemann–Liouville fractional derivative, subject to a couple of sets of boundary value conditions. In particular, we first consider problem (1.1) subject to

$$u(0) = u'(0) = u''(0) = u''(1) = 0. \tag{1.2}$$

We then consider the case in which the boundary value conditions are changed to

$$u(0) = u'(0) = u''(0) = 0, \quad u''(1) = \beta u''(\eta), \tag{1.3}$$

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where $\eta, \beta\eta^{\alpha-3} \in (0,1)$. The practical relevance of $3 < \alpha \leqslant 4$ appears in problems related with others areas as physics, economics which can be modeled by these fractional boundary value problems (see [17] and the references therein). Particularly, these problems appear in the Hamiltonian formulation for the Lagrangians depending on fractional derivatives of coordinates when the systems are non conservative (see, for example, [11]).

When $g(t,u) \equiv 0$, Liang and Zhang [16] investigated the existence of positive solutions for problem (1.1),(1.2) by means of lower–upper solution method and Schauder fixed-point theorem. They present the following result.

Theorem 1.1. ([16]). *Let* $\rho(t) = \frac{1}{\Gamma(\alpha)}\left(\frac{t^{\alpha-1}}{\alpha-2} - \frac{1}{\alpha}t^{\alpha}\right)$. *The fractional boundary value problem (1.1),(1.2) with* $g(t,u) \equiv 0$ *has a positive solution* $u(t)$ *if the following conditions are satisfied:*

(H_f) $f(t,u) \in C([0,1] \times [0,\infty), R^+)$ *is nondecreasing relative to u,* $f(t,\rho(t)) \not\equiv 0$ *for* $t \in (0,1)$ *and there exists a positive constant* $\mu < 1$ *such that*

$$k^{\mu} f(t,u) \leqslant f(t,ku), \quad \forall\ 0 \leqslant k \leqslant 1.$$

In [17], *by means of a fixed-point theorem in partially ordered sets, Caballero et al. considered the existence and uniqueness of positive solutions for fractional boundary value problem (1.1),(1.2) with* $g(t,u) \equiv 0$. *They present the following result.*

Theorem 1.2 ([17]). *Problem (1.1), (1.2) with* $g(t,u) \equiv 0$ *has a unique positive solution* $u(t)$ *if the following conditions are satisfied:*

(H_1) $f: [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and nondecreasing with respect to the second argument.
(H_2) There exists $t_0 \in [0,1]$ such that $f(t_0,0) > 0$.
(H_3) There exists $0 < \lambda \leqslant \frac{(\alpha-2)\Gamma(\alpha+1)}{2}$ such that, for $x,y \in [0,+\infty)$ with $y \geqslant x$ and $t \in [0,1]$,

$$f(t,y) - f(t,x) \leqslant \lambda \cdot \psi(y-x),$$

where $\psi : [0,\infty) \to [0,\infty)$*is continuous, nondecreasing, satisfying that* $Id - \varphi : [0,\infty) \to [0,\infty)$ *is continuous, nondecreasing, positive in* $(0,\infty)$ *and* $Id(0) = \varphi(0)$, *here Id denotes the identity mapping on* $[0,\infty)$.

By a similar method in [17], Liang and Zhang [21] studied a unique positive solution for problem (1.1) and (1.3) with $g(t,u) \equiv 0$. *They also present the following result.*

Theorem 1.3. ([21]). *Problem (1.1) and (1.3) with* $g(t,u) \equiv 0$ *has a unique positive and strictly increasing solution* $u(t)$ *if the following conditions are satisfied:*

(H_4) $f: [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing with respect to the second argument and* $f(t,u(t)) \not\equiv 0$ *for* $t \in Z \subset [0,1]$ *with* $\mu(Z) > 0$ *(*μ *denotes the Lebesgue measure);*
(H_5) *There exists* $0 < \lambda < \left(\frac{2}{(\alpha-2)\Gamma(\alpha+1)} + \frac{\beta\eta^{\alpha-3}(1-\eta)}{(\alpha-2)(1-\beta\eta^{\alpha-3})\Gamma(\alpha)}\right)^{-1}$ *such that for* $x,y \in [0,+\infty)$ *with* $y \geqslant x$ *and* $t \in [0,1]$,

$$f(t,y) - f(t,x) \leqslant \lambda \cdot \ln(y-x+1).$$

Our main interest in this paper is to give some alternative answers to the main results of these papers [16,17,21]. *We will use a fixed point theorem for a sum operator to show the existence and uniqueness of positive solutions for problems* (1.1) and (1.2) and (1.1) and (1.3). *Moreover, we can construct some sequences for approximating the unique solution.*

2. Preliminaries and previous results

For the convenience of the reader, we present here some definitions, lemmas and basic results that will be used in the proofs of our theorems.

Definition 2.1. ([4, Definition 2.1]). The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0$$

is called the Riemann–Liouville fractional integral of order α, where $\alpha > 0$ and $\Gamma(\alpha)$ denotes the gamma function.

Definition 2.2. ([4, pp. 36–37]). For a function $f(x)$ given in the interval $[0,\infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann–Liouville fractional derivative of order α.