Commun Nonlinear Sci Numer Simulat 19 (2014) 4037–4041

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Letter to the Editor

Generalization of the simplest equation method for nonlinear non-autonomous differential equations

Anastasia O. Antonova, Nikolay A. Kudryashov *

Department of Applied Mathematics, National Research Nuclear University "MEPhI", 31 Kashirskoe Shosse, 115409 Moscow, Russian Federation

ARTICLE INFO

Article history:
Received 3 December 2013
Accepted 4 March 2014
Available online 24 April 2014

Keywords:
Simplest equation method
Painlevé equation
Special solution
Riccati equation

ABSTRACT

It is known that the simplest equation method is applied for finding exact solutions of autonomous nonlinear differential equations. In this paper we extend this method for finding exact solutions of non-autonomous nonlinear differential equations (DEs). We applied the generalized approach to look for exact special solutions of three Painlevé equations. As ODE of lower order than Painlevé equations the Riccati equation is taken. The obtained exact special solutions are expressed in terms of the special functions defined by linear ODEs of the second order.

1. Introduction

Recently significant results were achieved in the development of methods to search for the exact solutions of nonlinear differential equations [1–21]. Such equations have been permanently appearing in different applied problems [22].

A lot of methods exist to look for exact solutions of nonlinear partially solvable differential equations, such as the singular manifold method [1,2], the tanh-function method [3–5], the G'/G-expansion method [6,7], the simplest equation method [11–19] and so on.

However, there is no systematic approach for finding the exact solutions of non-autonomous nonlinear differential equations. Meanwhile the non-autonomous DE are widely used in different applied problems and can appear when partial differential equations are solved. For instance, using the self-similar variables Korteweg-de Vries equation, modified Korteweg-de Vries equation, Burgers equation and others are reduced to non-autonomous ODEs. Moreover most of them can be converted to Painlevé equations.

For the first time the simplest equation method was presented in the work [11,12] to solve autonomous nonlinear DEs. There are several advantages of this method, namely: generalization of a lot of methods applied before, simplicity of realization and taking into account all possible singularities of equation solved.

In this Letter we extend the application of the simplest equation method to non-autonomous nonlinear ODEs. Three Painlevé equations were solved to show the application of this approach. The Riccati equation was chosen as ODE of lower order than Painlevé equations. The non-autonomy of the considered nonlinear ODE is reflected in functional dependence on independent variable in all coefficients existing in the Riccati equation and in substitution (2.2).

* Corresponding author. Tel.: +7 4993241181; fax: +7 4993239072.
E-mail addresses: nakudr@gmail.com, nakudryashov@mephi.ru (N.A. Kudryashov).

http://dx.doi.org/10.1016/j.cnsns.2014.03.035
1007-5704/

2. Method applied

Let us consider a nonlinear non-autonomous differential equation:

$$E[w, w', \dots, w^{(n)}, z] = 0. \tag{2.1}$$

Assuming ODE (2.1) possesses the first order singularity we shall search for exact solutions in the form as:

$$w(z) = f_1(z) + f_2(z)Y(z), \tag{2.2}$$

where $f_1(z),\ f_2(z)$ are functions which we have to find, $f_2(z) \neq 0$ identically. $Y(z)$ satisfies the Riccati equation:

$$Y_z(z) = a(z)Y^2(z) + b(z)Y(z) + c(z), \quad a(z) \neq 0. \tag{2.3}$$

The general solution of the Riccati equation (2.3) has the pole of the first order with respect to movable points, and, hence, function (2.2) has singularity of the same order.

The significant difference between extended method and approach presented in [12] consists in assumption that all coefficients in (2.2) and (2.3) depend on z. It takes into account non-autonomy of Eq. (2.1).

The algorithm reflecting the main idea of generalized simplest equation method is outlined below:

Step 1: the expressions (2.2) and (2.3) ought to be substituted in (2.1) and the coefficients of power of $Y(z)$ should be equating with zero. The solutions of the obtained equations are the expressions for $f_1(z),\ f_2(z),\ a(z),\ b(z),\ c(z)$ and for coefficients in Eq. (2.1).

Step 2: after *step 1* has been executed, the found expressions for coefficients in (2.2) and (2.3) are needed to be substituted in Riccati equation (2.3).

Step 3: Eq. (2.2) can be linearized by realization of two sequential substitutions [22]:

(a)

$$Y(z) = B_2(z) + B_1(z)R(z) \tag{2.4}$$

If $B_1(z) = -\frac{1}{a(z)},\quad B_2(z) = -\frac{a_z(z)+a(z)b(z)}{2a^2(z)}$, the Riccati equation (2.3) is transformed in

$$R_z(z) = -R^2(z) - q(z). \tag{2.5}$$

(b) $R(z) = \frac{U_z(z)}{U(z)}$, and Eq. (2.5) is converted in

$$U_{zz}(z) + q(z)U(z) = 0, \tag{2.6}$$

and the form of solution (2.2) is transformed in

$$w(z) = f_1(z) + f_2(z)B_2(z) + f_2(z)B_1(z)\frac{U_z(z)}{U(z)}. \tag{2.7}$$

After accomplishment of steps 1–3 for each of Eq. (2.1), the solutions presented in Section 3–5 have been obtained.

3. Exact solutions of the second Painlevé equation

Consider the second Painlevé equation:

$$w_{zz} = 2w^3(z) + zw(z) + \alpha. \tag{3.1}$$

Let us apply the algorithm expounded in Section 1 to find the solution of Eq. (3.1). Completing *step 1* we obtain equations for coefficients determination:

$$2f_2(z)(f_2(z) - a(z))(f_2(z) + a(z)) = 0; \tag{3.2}$$

$$3f_2(z)a(z)b(z) + 2f_{2_z}(z)a(z) - 6f_1(z)f_2^2(z) + f_2(z)a_z(z) = 0; \tag{3.3}$$

$$f_2(z)b^2(z) + 2f_{2_z}(z)b(z) + f_{2_{zz}}(z) - 6f_1^2(z)f_2(z) + 2f_2(z)a(z)c(z) - zf_2(z) + f_2(z)b_z(z) = 0; \tag{3.4}$$

$$f_{1_{zz}}(z) - 2f_1^3(z) + f_2(z)b(z)c(z) + 2f_{2_z}(z)c(z) - zf_1(z) + f_2(z)c_z(z) - \alpha = 0. \tag{3.5}$$

Solving sequentially Eqs. (3.2)–(3.4) we find coefficients $f_2(z),\ f_1(z),\ c(z)$ respectively.

$$f_2(z) = \pm a(z); \tag{3.6}$$