Commun Nonlinear Sci Numer Simulat 16 (2011) 101–111

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Positive solutions for second order impulsive differential equations with integral boundary conditions

Xinan Hao [a], Lishan Liu [a,b,*], Yonghong Wu [b]

[a] Department of Mathematics, Qufu Normal University, Qufu 273165, Shandong, People's Republic of China
[b] Department of Mathematics and Statistics, Curtin University of Technology, Perth, 6845 WA, Australia

ARTICLE INFO

Article history:
Received 17 October 2009
Accepted 8 April 2010
Available online 11 April 2010

Keywords:
Impulsive differential equations
Positive solution
Existence
Multiplicity
Integral boundary conditions

ABSTRACT

This paper is concerned with a class of second order impulsive differential equations with integral boundary conditions. Under different combinations of superlineary and sublinearity of nonlinear term and the impulses, various existence, multiplicity, and nonexistence results for positive solutions are derived in terms of the parameter lies in some intervals. The results obtained herein generalize and improve some known results.

1. Introduction

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years and has been developed very rapidly due to the fact that such equations find a wide range of applications modeling adequately many real processes observed in physics, chemistry, biology and engineering. Correspondingly, applications of the theory of impulsive differential equations to different areas were considered by many authors and some basic results on impulsive differential equations have been obtained, see [1–4] and the references therein.

In this paper we consider the existence, multiplicity, and nonexistence of positive solutions for the following singular integral boundary value problem with impulse effects

$$\begin{cases} u''(t)+a(t)u'(t)+b(t)u(t)+\lambda c(t)f(t,u(t))=0, & t\neq t_k,\ t\in J\setminus\{0,1\},\\ -\Delta u'|_{t=t_k}=\lambda I_k(u(t_k)), & k=1,2,\ldots,m,\\ u(0)=\int_0^1 g(s)u(s)ds, & u(1)=\int_0^1 h(s)u(s)ds, \end{cases} \tag{1.1}$$

where $J=[0,1],\ a\in C(J),\ b\in C(J,(-\infty,0)),\ c\in C((0,1),\mathbb{R}^+),\ c(t)\not\equiv 0$ is allowed to be singular at $t=0,1$. $g,h\in L^1[0,1]$ is nonnegative, λ is a positive parameter. Let $0=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=1$ be given, $f\in C(J\times\mathbb{R}^+,\mathbb{R}^+),\ I_i\in C(\mathbb{R}^+,\mathbb{R}^+),\ i=1,2,\ldots,m,\ \mathbb{R}^+=[0,+\infty)$. $\Delta u'|_{t=t_k}=u'(t_k^+)-u'(t_k^-)$, where $u'(t_k^+)$ (respectively $u'(t_k^-)$) denotes the right limit (respectively left limit) of $u'(t)$ at $t=t_k$.

Multi-point boundary value problem is a special case of the boundary value problem with integral boundary conditions. Problems with integral boundary conditions arise naturally in thermal conduction problems, semiconductor problems,

* Corresponding author at: Department of Mathematics, Qufu Normal University, Qufu 273165, Shandong, People's Republic of China. Tel.: +86 537 4458275; fax: +86 537 4455076.
E-mail addresses: haoxinan2004@163.com (X. Hao), lls@mail.qfnu.edu.cn (L. Liu), yhwu@maths.curtin.edu.au (Y. Wu).

1007-5704/$ - see front matter
doi:10.1016/j.cnsns.2010.04.007

hydrodynamic problems (see for instance [5–7]). Now such nonlocal boundary value problems of impulsive differential equations have not yet had great attention and have seldom been studied. This paper is to fill this gap in the literature.

For the case of $I_k = 0$, $k = 1,2,\ldots,m$, $\lambda = 1$, one of the special case of problem (1.1) is the following multi-point boundary value problem

$$\begin{cases} u''(t) + a(t)u'(t) + b(t)u(t) + c(t)f(u) = 0, & t \in (0,1), \\ u(0) = 0, & u(1) = \sum_{i=1}^{n} \alpha_i u(\xi_i), \end{cases} \tag{1.2}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$. Boundary value problem (1.2) and related problems have been extensively studied in many papers in recent years (see [8–14] and references therein). The existence and multiplicity results of positive solutions are obtained by applying the Krasnoselskii's fixed point theorem in cones, Leggett–Williams fixed point theorem and fixed point index theory. Naturally our work here will extend and unify some known results for nonlocal, especially multi-point, boundary value problems in the literature.

Recently, Lin and Jiang [15] studied the second order impulse boundary value problem

$$\begin{cases} -x''(t) = f(t,x), & t \neq t_k, \quad t \in (0,1), \\ -\Delta x'|_{t=t_k} = I_k(x(t_k)), & k = 1,2,\ldots,m, \\ x(0) = x(1) = 0. \end{cases} \tag{1.3}$$

The existence of multiple positive solutions is established via the theory of fixed point index in cones. Feng and Xie [16] dealt with the second order m-point boundary value problem with impulse effects

$$\begin{cases} -x''(t) = f(t,x), & t \neq t_k, \quad t \in (0,1), \\ -\Delta x'|_{t=t_k} = I_k(x(t_k)), & k = 1,2,\ldots,m, \\ x(0) = \Sigma_{i=1}^{m-2} a_i x(\xi_i), & x(1) = \Sigma_{i=1}^{m-2} b_i x(\xi_i), \end{cases} \tag{1.4}$$

where $a_i,\ b_i \in (0,1),\ 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ \Sigma_{i=1}^{m-2} b_i \xi_i < 1,\ \Sigma_{i=1}^{m-2} a_i(1-\xi_i) < 1$. The existence results of one and two positive solutions are obtained based on fixed point theorems in a cone.

Motivated by the papers mentioned above, in this paper, we consider the second order impulsive singular integral boundary value problem (1.1). Under different combinations of superlineary and sublinearity of nonlinear term f and the impulses I_k, $k = 1,2,\ldots,m$, various existence, multiplicity, and nonexistence results for positive solutions are derived in terms of different values of λ. Some ideas of this paper are from [17,18]. Our argument is based on Krasnosel'skii's fixed point theorem. Problem (1.1) is more general, it including two-point, multi-point, nonlocal and impulsive problems as special cases. Hence, we generalize and improve some known results in the literature to some degree, and so it is interesting and important to study the positive solutions for Problem (1.1).

The rest of this paper is organized as follows. In Section 2, we present some preliminaries and lemmas. The main results are stated in Section 3. In Section 4, we prove the main results.

2. Preliminaries and lemmas

In this section, we first introduce some background definitions in Banach spaces, state fixed point theorems, and then present basic lemmas that are very important in the proof of the main results.

Let $J' = J \setminus \{t_1, t_2, \ldots, t_m\},\ PC^1(J, \mathbb{R}^+) = \{u : u$ is a map from J into $\mathbb{R}^+$such that $u(t)$ is continuously differentiable at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+),\ u'(t_k^+), \ldots u'(t_k^-)$ exist, $k = 1,2,\ldots,m\}$. Then $PC^1(J, \mathbb{R}^+)$ is a real Banach space with norm $\|u\|_{PC^1} = \max\{\|u\|_{PC},\ \|u'\|_{PC}\}$, where

$$\|u\|_{PC} = \sup_{t\in J} |u(t)|, \quad \|u'\|_{PC} = \sup_{t \in J} |u'(t)|.$$

Notice that $PC(J, \mathbb{R}^+) = \{u : u$ is a map from J into $\mathbb{R}^+$such that $u(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and (t_k^+) exist, $k = 1,2,\ldots,m\}$ is also a Banach space with norm $\|u\|_{PC} = \sup_{t\in J}|u(t)|$. A function $u \in PC^1(J, \mathbb{R}^+) \cap C^2(J', \mathbb{R})$ is called a solution of problem (1.1) if it satisfies (1.1).

The proof of our main results is based upon an application of the following well-known Krasnosel'skii's fixed point theorem [19].

Lemma 2.1. *Let X be a Banach space, and P be a cone in X. Assume that Ω_1 and Ω_2 are two bounded open subsets of X with $0 \in \Omega_1,\ \overline{\Omega}_1 \subset \Omega_2$. Let $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator, satisfying either*

(i) $\|Ax\| \leqslant \|x\|, \quad x \in P \cap \partial\Omega_1, \quad \|Ax\| \geqslant \|x\|, \quad x \in P \cap \partial\Omega_2,$

or

(ii) $\|Ax\| \geqslant \|x\|, \quad x \in P \cap \partial\Omega_1, \quad \|Ax\| \leqslant \|x\|, \quad x \in P \cap \partial\Omega_2.$

Then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

In our main results, we will make use of the following lemmas.