Commun Nonlinear Sci Numer Simulat 16 (2011) 309–323

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Chaotic properties of the truncated elliptical billiards

V. Lopac [a,*], A. Šimić [b]

[a] Department of Physics, Faculty of Chemical Engineering and Technology, University of Zagreb, Croatia
[b] Department of Physics and Applied Mathematics, Faculty of Science, University of Navarra, Pamplona, Spain

ARTICLE INFO

Article history:
Received 28 January 2010
Accepted 16 March 2010
Available online 19 March 2010

Keywords:
Chaotic billiards
Truncated elliptical billiard
Elliptical stadium billiard
Poincaré sections
Box-counting method
Orbit stability
Chaotic fraction
Resonant cavities
Optical resonator
Hamiltonian system

ABSTRACT

Chaotic properties of symmetrical two-dimensional stadium-like billiards with piecewise flat and elliptical segments are studied numerically and analytically. Their sensitivity to small variations of the shape parameters can be usefully applied for optimal construction of the dielectric and polymer optical microresonators. For the two-parameter truncated elliptical billiards (TEB) the existence and linear stability of several periodic orbits are investigated in the full parameter space. Poincaré plots are computed and used for evaluation of the chaotic fraction of the phase space by means of the box-counting method. A highly chaotic behavior prevails in the region of elongated elliptical arcs, where most of the existing orbits are either neutral or unstable. In the parameter space, the upper limit of the fully chaotic behavior is reached when the relation between the two shape parameters becomes $\delta = \sqrt{1-\gamma^2}$, corresponding to truncated circles. Above this limit, for flattened elliptical arcs, mixed dynamics with numerous stable elliptic islands is found. In both regions parabolic orbits are present, many of them identical to orbits within an ellipse. These properties of the TEB differ remarkably from the behavior in the elliptical stadium billiards (ESB), where the chaotic region in the parameter space was strictly bounded from both sides. In order to follow the transition from TEB to ESB, a generalisation to a novel three-parameter family (GTEB) of stadium-like boundary shapes with elliptical arcs is proposed.

1. Introduction

Two-dimensional planar billiards are nonlinear Hamiltonian systems with rich and interesting dynamical properties. In the classical mechanics, a point particle, moving with constant velocity within a closed boundary and exhibiting specular reflections on the walls, can have regular, mixed or fully chaotic dynamics. This behavior depends strongly on the shape of the billiard boundary, and the resulting dynamical properties are extremely sensitive to small variations of the shape parameters. A detailed exposition of the fundamental properties of classical chaotic billiards, including a list of relevant references, is given in the book by Chernov and Markarian [1].

With exception of some special classes of polygonal billiards, the only known boundary shape with integrable classical dynamics is ellipse, with the circular billiard included [2]. Complete classical solutions for the family of confocal elliptical billiards, including also the full elliptical shapes, have been explored in [3]. The fully chaotic billiards are the Sinai billiard [4], the Bunimovich stadium billiard [5] and the cardioid (Robnik) billiard [6–8]. The Sinai billiard is dispersing, predictably leading to chaotic dynamics. However, the stadium billiard, with a rectangle inserted between the two half-circles, is also chaotic. It has been found that the defocusing mechanism responsible for chaos in the circular stadium is present also in a number of other shapes and different billiard shapes with fully chaotic dynamics have been constructed [9–12]. Interpretations of mechanisms responsible for creating chaoticity in billiards, explaining how and why the absolutely focusing

* Corresponding author. Tel.: +385 1 45 97 106; fax: +385 1 45 97 135.
E-mail address: vlopac@marie.fkit.hr (V. Lopac).

1007-5704/$ - see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cnsns.2010.03.007

components may generate hyperbolicity in billiards, containing the most recent results, are reviewed in [13]. Much attention has been paid to numerical investigations of billiards with mixed dynamics, especially of those with boundaries consisting of circular arcs, but also with elliptical, hyperbolic and parabolic segments [2,14–20]. A two-dimensional planar billiard can also be approached as a quantum system, since the corresponding Schrödinger equation is mathematically equivalent to the Helmholtz equation for the vibrations of a two-dimensional membrane. The quantal energy spectra and wave functions, reflecting the chaotic properties of the corresponding classical billiards, are equally sensitive to the shape variations [21–26]. Importance of the billiard shapes has been stressed also in the recent investigations of the graphene quantum dots [27].

All these properties place billiards among the most attractive types of dynamical models in the study of ergodic and mixing properties of Hamiltonian systems [28,2]. The recent revival of general interest for billiards is mostly due to the fact that there are many physical systems whose study can be reduced to a billiard-type problem. The problem of stationary electromagnetic waves within the waveguides and optical cavities in the limit of small wavelengths can be approximated by the ray propagation mechanism, which is mathematically equivalent to the problem of a point particle moving freely in a billiard. In applications to laser technology, the consideration of chaotic dynamics was crucial for increasing the output power of the dielectrical, semiconducting and polymer microlasers [29]. In experiments with the asymmetric resonant cavities, the critical element in technological improvements has been provided either by distorting slightly the circular shape of the optical resonators [30,31] or by considering the long and narrow stadium and quasi-stadium shapes [32–34].

The fine adjustments of the resonator shapes described above can be achieved by using the two-parameter planar boundary with two symmetrically placed elliptical arcs at the opposite sides of a rectangle. In our previous work we investigated the two-parameter elliptical stadium billiard (ESB), first explored by Donnay [10], which is also a special case of the mushroom billiard [11,35–37]. The upper limit of the fully chaotic region in the parameter space for ESB was found by Wojtkovski [9]. Our numerical investigation in the full parameter space has shown that the ESB are fully chaotic for a bounded set of shape parameters and have mixed behavior in other parameter regions [17,38]. This confirmed the results of analysis [39–42] identifying the emergence of the stable pantographic orbits as the mechanism generating the lower limit of chaos in the elliptical stadium billiards. In this work, we extend our investigations to a larger class of stadium-like billiards. We first examine the dynamical properties of the truncated elliptical billiards (TEB), another family of billiards with elliptical arcs joined by flat segments. We follow the procedure used in [17] for elliptical stadia (ESB) and compare the results for these two billiard families, as a possible basis for estimating the directional output power from the corresponding dielectrical cavities, in dependence on the resonator shapes [43].

The TEB is defined as a point particle in the two-parameter planar domain, constructed by truncating an ellipse on opposite sides (Fig. 1). A symmetrical stadium-like shape thus obtained consists of a rectangle with two elliptical arcs added at its opposite ends. The corresponding billiard has been analysed by Del Magno [44] who, investigating a restricted part of the parameter space and applying the method of invariant cones, determined the region of hyperbolic behavior and presented an estimate of the region where such billiard could be ergodic. Although chaotic dynamics of the truncated elliptical billiard was not analysed elsewhere, the shape itself has been introduced in different contexts: as a particular cross-section of the three-dimensional cavity, for example in description of the barrel billiard introduced by Bunimovich [45,46], or as a geometrical model for the mathematical description of the human heart in [47]. The chaos in billiards obtained by various ways of truncating the circle has been considered in [48–50].

In our approach, the two billiard families, TEB and ESB, have in common the two shape parameters δ and γ, which determine half of the width and the height, respectively, of the central rectangle. They both include as limits the circular, elliptical, rectangular and quadratic shapes. However, whereas the ESB comprises the Bunimovich (circular) stadium, this shape does not exist within the TEB. For computational reasons, the horizontal diameters of these billiards were normalised to 2, in the same way as in our previous work on symmetrical lemon-shaped billiards with the parabolic, generalised power-law, hyperbolic and elliptical arcs [18–20].

In this paper, we investigate numerically and analytically the TEB in the full parameter space, described by two parameters δ and γ. In Section 2, we define the TEB boundary and describe its geometrical properties. In Section 3, the existence and stability conditions of selected periodic orbits are discussed and illustrated by Poincaré plots and orbit diagrams. In Section 4, the Poincaré sections are used to compute, with the box-counting method, the chaotic fraction of the phase space, for a large set of shape parameters. The region in which the billiard seems to be fully chaotic is identified, and for the remaining part of the parameter space the numerical estimates of the chaotic fraction, in dependence on the shape, are presented. The results are shown in the parameter-space diagram and compared with the same type of diagram for the elliptical stadium billiard. In Section 5, we propose a possible generalisation (GTEB) of the truncated elliptical billiard, providing a transition between the two types of the stadium-like elliptical billiards ESB and TEB. In Section 6, we summarise the obtained results and propose further investigations. At the end, in a brief Appendix A, we discuss the box-counting method and analyse the dependence of results on relevant parameters.

2. Geometrical properties of the truncated elliptical billiard

In our parametrisation the TEB is defined in the (x, y) plane by means of the two shape parameters δ and γ, satisfying conditions $0 \leqslant \delta \leqslant 1$ and $0 < \gamma < \infty$. The billiard boundary is described as