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Chaos in rational systems in the plane containing quadratic terms

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ABSTRACT

Consider the following system of rational equations containing quadratic terms

$$\begin{cases} x_{n+1} = \frac{A_1x_n^2+B_1x_ny_n+C_1y_n^2+D_1x_n+E_1y_n+F_1}{\alpha_1x_n^2+\beta_1x_ny_n+\gamma_1y_n^2+\lambda_1x_n+\mu_1y_n+\nu_1}, \\ y_{n+1} = \frac{A_2x_n^2+B_2x_ny_n+C_2y_n^2+D_2x_n+E_2y_n+F_2}{\alpha_2x_n^2+\beta_2x_ny_n+\gamma_2y_n^2+\lambda_2x_n+\mu_2y_n+\nu_2}. \end{cases}$$

Chaos in the sense of Li–Yorke is considered. This is based on the Marotto's theorem via obtaining a snap-back repeller. In fact, first in a special case when $F_1 = F_2 = 0$, we show that origin is a snap-back repeller and so the system has chaotic behavior in the sense of Li–Yorke under some conditions. Then in a more general case, we prove that existence of chaos in the sense of Li–Yorke for the above system.

1. Introduction

Recently study on the rational systems having both a linear numerator and a linear denominator have started extensively by Gerry Ladas and colleagues; see e.g. [1–3,12–14,17,20–22] which include extensive lists of references. This substantial research makes a strong case for studying the behaviour of rational difference equations as well as providing a great deal of information about the behaviour of rational equations with linear terms. But there is not a systematically study on the rational systems having quadratic terms about all of dynamics specifically chaotic dynamics. A class of rational systems in the plane with quadratic terms include systems such that are reducible to some second order rational difference equation with quadratic terms. Route to chaos in a class of second order rational difference equation with quadratic terms is studied in [7]. Applications of rational equations containing quadratic and cubic terms to biological models have been discussed in Refs. [5,6] via systems of first order rational equation using monotone methods. As a famous chaotic two-dimensional rational system having quadratic terms in the polynomial form with a strange attractor is the Henon system

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + y_n, \\ y_{n+1} = bx_n, \end{cases}$$

where a and b are real parameters with $|b| < 1$ (see [8–10]). Also chaos in the sense of Devaney for polynomial quadratic systems, which is a special case of the rational systems considered in [4,23]. In this manuscript, chaos in the sense of Li–Yorke for rational systems having quadratic terms in the both numerator and denominator i.e. the systems defined in the abstract is considered. The definition of chaos in the sense of Li and Yorke is now summarized.

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Definition 1.1 (Li and Yorke [16]). Let I be an interval of the real line and $f: I \to I$ a continuous function. f is chaotic in the sense of Li–Yorke if f has an uncountable scrambled set $S \subset I$ which satisfies the following condition:

(i) for every $x_S, y_S \in S$ with $x_S \neq y_S$,

$$\limsup_{n\to\infty} |f^n(x_S) - f^n(y_S)| > 0, \quad \text{and} \quad \liminf_{n\to\infty} |f^n(x_S) - f^n(y_S)| = 0,$$

(ii) for every $x_S \in S$ and any periodic point y_{per} of f

$$\limsup_{n\to\infty} |f^n(x_S) - f^n(y_{per})| > 0.$$

Li and Yorke defined only chaos on an interval, Huang and Ye extended the interval to general metric space.

Definition 1.2 [11]. Let $f: X \to X$ is continuous and surjective, where X is a compact metric space with metric d. A subset $S \subset X$ is a scrambled set (for f), if any different points x_S and y_S from S are proximal and not asymptotic:

$$\limsup_{n\to\infty} d(f^n(x_S), f^n(y_S)) > 0, \quad \text{and} \quad \liminf_{n\to\infty} d(f^n(x_S), f^n(y_S)) = 0.$$

The function f is said to be chaotic in the sense of Li–Yorke, if there exists an uncountable scrambled set.

A very useful tool in proving chaotic behavior in an $k-$ dimensional discrete dynamical systems is the well-known Marotto's theorem in [18]: "snap-back repellers imply chaos in $\mathbb{R}^k$". Marotto's chaos is the modified definition of chaos in the sense of Li–Yorke for multidimensional discrete dynamical systems. But several authors, most recently Li and Chen in [15], pointed out that there is an error in the original proof of Marotto, and tried to correct this error giving new, weaker versions of the theorem. Marotto in [19] corrected the technical flaw in the original snap-back repeller theorem, obtaining the following result (which is the most general result known so far):

Theorem 1.3 [18]. *Let $f \in C^1(\mathbb{R}^k, \mathbb{R}^k)$ has a snap-back repeller z^*, i.e.*

(a) *z^* is a fixed point of f;*
(b) *all eigenvalues of $Df(z^*)$ exceed 1 in magnitude;*
(c) *there exists a point $z_0 \neq z^*$ in a repelling neighborhood of z^*, such that $z_M = z^*$ and $det(DF(z_j)) \neq 0$ for $j = 1, \ldots, M$, where $z_j = f^j(z_0)$ (for some integer M).*

Then f is chaotic in the sense of Li–Yorke.

First remember that all of the roots of the quadratic polynomial

$$\lambda^2 - p\lambda - q = 0,$$

are outside the unit disc iff the following conditions hold:

(i) $|q| > 1$;

(ii) $|p| < |1 - q|$.Also note that according to general formulas for roots of quadratic polynomials, the magnitude of roots are continuous functions of coefficients. This notation is useful for proving the chaos in next section.

2. Case $F_1 = F_2 = 0$,

Consider the system of rational equations

$$\begin{cases} x_{n+1} = \frac{A_1x_n^2 + B_1x_ny_n + C_1y_n^2 + D_1x_n + E_1y_n + F_1}{\alpha_1x_n^2 + \beta_1x_ny_n + \gamma_1y_n^2 + \lambda_1x_n + \mu_1y_n + \nu_1}, \\ y_{n+1} = \frac{A_2x_n^2 + B_2x_ny_n + C_2y_n^2 + D_2x_n + E_2y_n + F_2}{\alpha_2x_n^2 + \beta_2x_ny_n + \gamma_2y_n^2 + \lambda_2x_n + \mu_2y_n + \nu_2}. \end{cases} \tag{2.1}$$

It is convenient in what follows to define the mapping $f: \mathbb{R}^2 \to \mathbb{R}^2$, such that $f(x, y) = (f_1(x, y), f_2(x, y))$, where

$$f_1(x, y) = \frac{A_1x^2 + B_1xy + C_1y^2 + D_1x + E_1y + F_1}{\alpha_1x^2 + \beta_1xy + \gamma_1y^2 + \lambda_1x + \mu_1y + \nu_1}, \quad f_2(x, y) = \frac{A_2x^2 + B_2xy + C_2y^2 + D_2x + E_2y + F_2}{\alpha_2x^2 + \beta_2xy + \gamma_2y^2 + \lambda_2x + \mu_2y + \nu_2}.$$

We consider solutions not on the forbidden set i.e. every solution (x_n, y_n) with

$$\alpha_ix_n^2 + \beta_ix_ny_n + \gamma_iy_n^2 + \lambda_ix_n + \mu_iy_n + \nu_i \neq 0, \quad n = 0, 1, 2 \ldots, \quad i = 1, 2.$$

Let $F_1 = F_2 = 0$. In this case the origin is a fixed point of f.

Lemma 2.1. *Let $F_1 = F_2 = 0$. All eigenvalues of $Df(0, 0)$ exceed 1 in magnitude iff*