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Mathematical modelling and symbolic dynamics analysis of three new Galton board models

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ABSTRACT

A Galton board, also known as a quincunx, is a device invented by Francis Galton in 1873 that consists of two upright boards with rows of pins, and a funnel. In this paper, three new mathematical models of Galton board that are of increasing complexity are formulated. The discussion includes a brief literature review, the description of the systems, the important physical processes, the assumptions employed and the derivation of the governing equations of the models. The quincunx models are folded into a discrete-time deterministic dynamical system, called the quincunx maps, that enables a simplified analysis of the symbolic dynamics. While Galton and countless subsequent statisticians have suggested that a small ball falling through a quincunx would exhibit random walk; the results of the symbolic dynamics analysis demonstrate that this is not the case. This paper presents evidence that the details of the deterministic models are not essential for demonstrating deviations from the statistical models.

1. Introduction

Francis Galton (1822–1911) was a tropical explorer, meteorologist, geneticist and statistician. He invented mechanical devices and conducted experiments, to demonstrate important statistical principles and to gain insights into areas of scientific interest. One of his inventions is the eponymous Galton board.

A Galton board (also called a quincunx) is a mechanical device which consists of two upright, parallel plates, a smooth wooden board and a glass sheet, with many interleaved horizontal rows of equally spaced pins. The pins are arranged on a board in a hexagonal array, as shown in Fig. 1. At the top of the quincunx device, there is a funnel into which small lead balls are released. At the bottom of the device there is a row of narrow rectangular compartments or bins where the balls are collected. The whole installation is covered with a glass sheet from the front to allow viewing. Galton claimed that the Galton board is a random system in which each ball has an equal probability of going to either side of every pin that it strikes, giving rise to a bell-shaped distribution of lead. That is, the final exit distribution of the balls at the bottom of the device approximates a Binomial or Gaussian distribution. Nowadays, it is often a classroom and textbook demonstration of probability theory, Brownian motion and statistical mechanics. However, most of the statistical-mechanical work involved purely-elastic collisions, followed by isokinetic trajectories [1,2]. Numerous animations and simulations of the Galton boards which can

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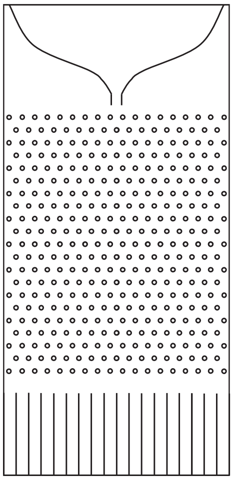

Fig. 1. Schematic of a Galton board.

be found on the internet are simply simulations of random walk or a Binomial process, not simulations of the device itself. A more detailed discussion on the Galton board can be found in the literature [3–6].

1.1. Previous Galton board models and related studies

There are several models of Galton board that have been developed for different purposes and using different sets of assumptions. Hoover and Moran (1992) conducted numerical simulations of the Galton board using isokinetics equations of motion. It was shown that the phase-space distribution of the isokinetics dynamical system showed a strange attractor or complex structure [7]. Lue and Brenner (1993) theoretically studied several aspects of the Galton board including the phase-flow dynamics of the system, Poincare map dynamics, numerical computation of the dynamical system, and the statistical structure of the Galton board systems [8]. Bruno et al. (2003) investigated Galton boards using physical experiments with polystyrene balls and numerical simulations. They suggested that certain dynamics and properties of the flow in a Galton board will give some insights in understanding granular mixing [9]. Kozlov and Mitrofanova (2003) investigated the properties of the balls' distribution over the compartments of the Galton board. They studied the dependency of the variance of this distribution on three parameters: the coefficient of the restitution; the pin's radius; the variance of the normal distribution of the initial condition [10]. Rosato et al. (2004) investigated the motion of a single particle travelling down a Galton board under the influence of gravity. The study was conducted using numerical simulations, physical experiments and using the simple dynamical models for certain ranges of important parameters [11]. Judd (2007) introduced the assumption that if the magnitude of the rebound velocity is less than a certain threshold value, a ball will stick to the pin then roll off without slipping [12]. Chernov and Dolgopyat (2008) considered the pins as convex obstacles which are positioned periodically on the board and satisfy the finite horizon condition, that is, the ball cannot move in any direction indefinitely without meeting a pin [13]. More information on these models can be found in [14].

The models have several common assumptions. For examples, the Galton board is modelled as a two-dimensional model, the collisions between the balls and the pins are considered inelastic (except in [13]) but instantaneous, and there is no rotational motion or angular velocity of the balls.

There are a few studies related to the Galton board. One example is the bouncing-ball problem, in which a ball collides inelastically with a sinusoidally moving table. The problem which has been studied extensively [15–17], is regarded as one of the simplest physical systems that can produce chaotic motion [18]. It has an element in common with the Galton board problem, in that it involves the inelastic impacts of balls between the free flights of balls. Another example is the Lorentz gas, which is similar to a purely elastic, infinite Galton board when gravity is absent. Furthermore, previous studies have employed numerical simulations of the Galton board as transport phenomena and percolation models [7,19]. These possible applications will not be explicitly examined in this paper but are offered only as motivation and background for studying the Galton board.

This study has three aims. The first aim is to construct plausible models of a Galton board with reasonably realistic assumptions about all the important physical processes. The second aim is to establish if a random walk is a good model for the developed quincunx model. That is, to address the question of whether the Galton board behaves in the way Galton envisaged. The third aim is to investigate whether the details of the deterministic models are essential for demonstrating deviations from the statistical models.

The organisation of the paper is as follows. Section 2 offers a detailed description of the characteristics, physical assumptions and governing equations of the quincunx models in this paper. Section 3 presents the algorithm and flowchart