Commun Nonlinear Sci Numer Simulat 19 (2014) 3600–3610

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

A partial Hamiltonian approach for current value Hamiltonian systems

R. Naz [a,*], F.M. Mahomed [b], Azam Chaudhry [c]

[a] Centre for Mathematics and Statistical Sciences, Lahore School of Economics, Lahore 53200, Pakistan
[b] Differential Equations, Continuum Mechanics and Applications, School of Computational and Applied Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa
[c] Department of Economics, Lahore School of Economics, Lahore 53200, Pakistan

ARTICLE INFO

Article history:
Received 11 June 2013
Accepted 21 March 2014
Available online 5 April 2014

Keywords:
Current value Hamiltonian
Partial Hamiltonian approach
Economic growth models

ABSTRACT

We develop a partial Hamiltonian framework to obtain reductions and closed-form solutions via first integrals of current value Hamiltonian systems of ordinary differential equations (ODEs). The approach is algorithmic and applies to many state and costate variables of the current value Hamiltonian. However, we apply the method to models with one control, one state and one costate variable to illustrate its effectiveness. The current value Hamiltonian systems arise in economic growth theory and other economic models. We explain our approach with the help of a simple illustrative example and then apply it to two widely used economic growth models: the Ramsey model with a constant relative risk aversion (CRRA) utility function and Cobb Douglas technology and a one-sector AK model of endogenous growth are considered. We show that our newly developed systematic approach can be used to deduce results given in the literature and also to find new solutions.

1. Introduction

There has been extensive use of dynamic optimization in economic modeling and many of these models use the current value Hamiltonian whenever the integrand function contains a discount factor. These models range from those used for neoclassical economic growth [1,2] to optimal firm-level investment [3] and human capital and earnings [4]. Pontrygin's maximum principle provides a set of necessary conditions for the solution of the continuous time optimal control problem involving a current value Hamiltonian and a dynamical system of ODEs is obtained for control, state and costate variables. Beginning with [5] there have been various approaches, both qualitative and quantitative (see [6] for a good account of these), to deal with dynamic economic models arising from current value Hamiltonian system and most of these models were solved using numerical approaches (like [7]) or linear approximations around steady states [8]. The critical problem is that for the underlying nonlinear dynamical system in economics there is a lack of a general analytical solution procedure not only for higher order systems but even for systems with one state and costate variable.

It is true to say that nonlinear dynamical systems evade closed-form solutions in general. However, the lack of a general procedure inhibits the search for reductions and solutions of such type of nonlinear equations even when solutions do exist. Having said that, there are some well-known closed-form solutions that appear in the literature (see, e.g. [9–14]). These

* Corresponding author. Tel.: +92 3315439545.
E-mail address: drrehana@lahoreschool.edu.pk (R. Naz).

http://dx.doi.org/10.1016/j.cnsns.2014.03.023
1007-5704/

solutions have been obtained by seemingly disparate approaches. Independent of the knowledge of explicit solutions, dynamic local stability of certain systems (see [15–17]) have been characterized by qualitative or numerical approaches.

Several important contributions have been made in the analysis of nonlinear dynamical systems of economic models. Here we focus on a new approach which yields first reductions and closed-form solutions for such systems of ODEs. We develop a Hamiltonian framework for several control, state and costate variables. Therefore, the method we develop is applicable to an arbitrary system of ODEs. However, we apply it to a system of two ODEs in order to show its effectiveness. In the case of higher order systems of ODEs, the approach may require the use of algebraic computing.

The layout of the paper is as follows. The partial Hamiltonian approach is developed in Section 2. In Section 3 we provide a simple illustrative example to show how our approach works. The Ramsey model with a constant relative risk aversion (CRRA) utility function with Cobb Douglas technology and the one-sector AK model of endogenous growth are studied in Section 4 and known solutions are deduced via our partial Hamiltonian approach. Conclusions are finally presented in Section 5.

2. A Hamiltonian version of the Noether-type theorem

Herein we develop a partial Hamiltonian approach for current value Hamiltonians which do not satisfy the canonical Hamilton equations. This is done for several control, state and costate variables.

Let t be the independent variable and $(q,p)=(q^1,\dots,q^n,p_1,\dots,p_n)$ the phase space coordinates. The derivatives of q^i, p_i with respect to t are

$$\dot{p}_i=D(p_i),\quad \dot{q}_i=D(q_i),\quad i=1,2,\dots,n, \tag{1}$$

where

$$D=\frac{\partial}{\partial t}+\dot{q}_i\frac{\partial}{\partial q_i}+\dot{p}_i\frac{\partial}{\partial p_i}+\cdots \tag{2}$$

is the total derivative operator with respect to t. The summation convention is utilized for repeated indices. The variables t, q, p are independent and connected only by the differential relations (1).

There are some well-known operators which are defined in the space of the variables (t,q,p) and its prolongations. We introduce them.

In addition to the Euler operator

$$\frac{\delta}{\delta q^i}=\frac{\partial}{\partial q^i}-D\frac{\partial}{\partial \dot{q}^i},\quad i=1,2,\cdots,n, \tag{3}$$

one also has the variational operator

$$\frac{\delta}{\delta p_i}=\frac{\partial}{\partial p_i}-D\frac{\partial}{\partial \dot{p}_i},\quad i=1,2,\cdots,n. \tag{4}$$

The action of the operators (3) and (4) on

$$L(t,q,\dot{q})=p_i\dot{q}^i-H(t,q,p), \tag{5}$$

equated to zero yields the canonical Hamilton equations

$$\dot{q}^i=\frac{\partial H}{\partial p_i},\quad \dot{p}_i=-\frac{\partial H}{\partial q^i},\quad i=1,\dots,n. \tag{6}$$

That is $\delta L/\delta q^i=0$ and $\delta L/\delta p_i=0$ results in (6). Eq. (5) is the well-known Legendre transformation which relates the Hamiltonian and Lagrangian, where $p_i=\partial L/\partial \dot{q}^i$ and $\dot{q}^i=\partial H/\partial \dot{p}_i$.

Generators of point symmetries in the space (t,q,p) are operators of the form

$$X=\xi(t,q,p)\frac{\partial}{\partial t}+\eta^i(t,q,p)\frac{\partial}{\partial q^i}+\zeta_i(t,q,p)\frac{\partial}{\partial p_i}. \tag{7}$$

The operator in (7) is a generator of a point symmetry of the canonical Hamiltonian system (6) if [18]

$$\dot{\eta}^i-\dot{q}^i\dot{\xi}-X\left(\frac{\partial H}{\partial p_i}\right)=0,\quad \dot{\zeta}_i-\dot{p}_i\dot{\xi}+X\left(\frac{\partial H}{\partial q^i}\right)=0,\quad i=1,\dots,n \tag{8}$$

on the system (6).

Hamiltonian symmetries in evolutionary or canonical form have been considered [18]. Furthermore, symmetry properties of the Hamiltonian action have been investigated in the space (t,q,p) by [19,20]. In the latter, the authors considered the general form of the symmetries (7) and provided a Hamiltonian version of Noether's theorem.

The following important results which are analogs of Noether symmetries and the Noether theorem (see [18,21,22,20] for a discussion) were established.