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Hyperchaotic set in continuous chaos–hyperchaos transition

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ABSTRACT

Topological horseshoes with two-directional expansion imply invariant sets with two positive Lyapunov exponents (LE), which are recognized as a signature of hyperchaos. However, we find such horseshoes in two piecewise linear systems and one smooth system, which all exhibit chaotic attractors with one positive LE. The three concrete systems are the simple circuit by Tamaševičius et al., the Matsumoto–Chua–Kobayashi (MCK) circuit and the linearly controlled Lorenz system, respectively. Substantial numerical evidence from these systems suggests that a hyperchaotic set can be embedded in a chaotic attractor with one positive LE, and keeps existing while the attractor becomes hyperchaotic from chaotic. This paper presents such a new scenario of the continuous chaos–hyperchaos transition.

1. Introduction

Hyperchaos, introduced by Rössler [1], is typically defined as chaotic behavior with more than one positive Lyapunov Exponent (LE), indicating that dynamics expands multi-directionally. So it is believed that hyperchaos is much more complex than chaos with only one positive LE (or just chaos for simplicity in the present paper) and has better application value in chaos engineering. Although hyperchaos has been studied for many years, its mechanism is still not well understood.

How a chaotic attractor becomes hyperchaotic, i.e., chaos–hyperchaos transition, is an interesting problem in nonlinear dynamics. Such transition has been observed widely in numerous hyperchaotic systems, such as electronic circuits, coupled oscillators, etc. Up to now, the transition has often been explained by blowout bifurcation [2], which means sudden change of dimension of unstable manifolds of a large number of unstable periodic orbits (UPO) embedded in the chaotic attractor. This indicates a clear boundary between chaotic attractors and hyperchaotic attractors, where the expansion suddenly becomes multi-directional from one-directional.

However, numerical studies show that attractors' dimension and the second LE grow continuously during the transition [3–9]. Such examples can be found in many different concrete systems, such as the simple 4-Dimensional (4D) chaotic electronic circuit proposed by Tamaševičius et al. [4], the linearly controlled Lorenz system by Yang, Zhang and Chen [8] and the hyperchaotic Matsumoto–Chua–Kobayashi (MCK) circuit [9]. In these systems, the second LE becomes positive "smoothly" as parameters vary, and the shape of attractor does not change much during the transition. We cannot find obvious change of dimension of unstable manifolds of UPO embedded in the attractors in the process of transition. This continuous transition

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from chaos to hyperchaos seems different from any other routes known before in [10–14]. Up to now, why the transition is "smooth" remains a mystery.

The existence of a horseshoe embedded in a dynamical system is probably the most compelling signature of chaos or hyperchaos [15,16]. It indicates that there exists a cantor-like invariant set in the phase space where the dynamics is (semi-)conjugate to an m-shift map. In chaos, the horseshoe only expands in one direction corresponding to one positive LE. In hyperchaos, the horseshoe expands at least in two directions [17–19]. Consequently, each trajectory in the invariant set have at least two positive LE, i.e., the set is hyperchaotic. After the chaos–hyperchaos transition, there is a hyperchaotic invariant set embedded in a hyperchaotic attractor. An interesting question is how the hyperchaotic set appears during the transition.

This paper studies a "smooth" chaos–hyperchaos transition by virtue of topological horseshoe theory. We will investigate three typical systems, i.e., the simple 4D circuit, the hyperchaotic Lorenz system and the hyperchaotic MCK circuit, and report an interesting phenomenon unknown before: hyperchaotic invariant sets can be embedded in chaotic attractors (i.e., chaotic attractors with one positive LE) and keep existing during the transition. This not only suggests that the phenomenon typically exists in the "smooth" chaos–hyperchaos transition, but may also present a scenario for the continuous transition from chaos to hyperchaos.

The rest of paper is organized as follows: Section 2 focuses on the simple 4D circuit, obtains a hyperchaotic invariant set while the circuit is chaotic, and shows that the set keeps existing during the "smooth" chaos–hyperchaos transition; Section 3 presents two more examples about such phenomenon with the hyperchaotic Lorenz system and the hyperchaotic MCK circuit; Section 4 draws conclusions.

2. An invariant hyperchaotic set in the simple 4D circuit

This section focuses on the simple 4D circuit described by [4]

$$\begin{cases} \dot{x}_1 = \alpha x_1 - x_2 - x_3 \\ \dot{x}_2 = x_1 \\ \dot{x}_3 = \gamma(x_1 - x_4) \\ \dot{x}_4 = \epsilon[x_3 - \beta(x_4 - 1)h(x_4 - 1)] \end{cases} \tag{1}$$

where $\beta = 10$, $\gamma = \epsilon = 3$ and $h(\cdot)$ is the Heaviside step function. Tamaševičius et al. found that system (1) only demonstrates one attractor as α varies. The LE at $\alpha = 0.4$ are 0.060, 0.00, -0.021 and -6.52. Only the first LE is positive, so the attractor should be chaotic, as shown in Fig. 1. The LE at $\alpha = 0.5$ are 0.065, 0.026, 0.00 and -5.98, where two of them are positive, so the attractor becomes hyperchaotic. There exists chaos–hyperchaos transition during $\alpha \in [0.4,\ 0.5]$.

According to the definition of LE, each positive LE indicates a direction of expansion of some attractor in the phase space, so the number of positive LE equals the dimension of expansion. However, although the attractor at $\alpha = 0.4$ has only one positive LE in system (1), we surprisingly find a topological horseshoe with two-directional expansion.

In the following, we first present a computation method, demonstrate existence of a hyperchaotic invariant set, then we consider the relation between the horseshoe and the chaotic attractor, finally we show that the hyperchaotic set keeps existing during the chaos–hyperchaos transition.

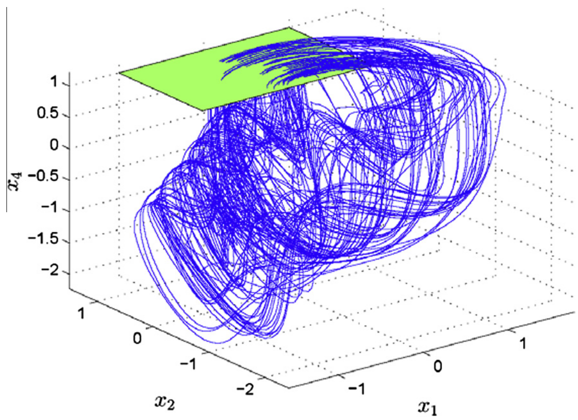

Fig. 1. The phase plot of (1) at $\alpha = 0.4$, $\beta = 10$ and $\gamma = \epsilon = 3$.