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Stabilization via parametric excitation of multi-dof statically unstable systems

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ABSTRACT

The problem of re-stabilization via parametric excitation of statically unstable linear Hamiltonian systems is addressed. An n-degree-of-freedom dynamical system is considered, at rest in a critical equilibrium position, possessing a pair of zero-eigenvalues and $n-1$ pairs of distinct purely imaginary conjugate eigenvalues. The response of the system to a small static load, making the zero eigenvalues real and opposite, simultaneous to a harmonic parametric excitation of small amplitude, is studied by the Multiple Scale perturbation method, and the stability of the equilibrium position is investigated. Several cases of resonance between the excitation frequency and the natural non-zero frequencies are studied, calling for standard and non-standard applications of the method. It is found that the parametric excitation is able to re-stabilize the equilibrium for any value of the excitation frequencies, except for frequencies close to resonant values, provided a sufficiently large excitation amplitude is enforced. Results are compared with those provided by a purely numerical approach grounded on the Floquet theory.

1. Introduction

It is well-known that when a Hamiltonian system resting at a stable equilibrium position is parametrically excited, depending on the frequency and on the amplitude of the excitation, it can lose stability, so that oscillations of large amplitude are triggered. However, the converse is also true, i.e. a Hamiltonian system at a (statically) unstable equilibrium point (i.e. at a divergence point) can be re-stabilized by a suitable parametric excitation! The phenomenon is appealing, since, if suitable exploited, it could suggest control strategies alternative to that discussed, e.g., in [1,2].

An example of such strategies is offered by the time-periodic control gain (see, e.g. [3,4], where the so-called act-and-wait control is studied). Here, the gains play the role of time-periodic spring and damper in mechanical systems, and therefore is closely related to the topic analyzed here. A short overview of various applications of stabilization by vibration, along with the exposition of the related geometrical mechanisms, can be found in [5].

A famous example of re-stabilization is represented by "the Indian magic rope trick problem" (see [6–8]), for which a vertical rope under self-weight, which would be statically unstable, is instead stable when the lower-end is driven to execute a vertical harmonic motion at a frequency higher than the resonant value. The phenomenon has been explained in literature (see, e.g. [8–10]) via the concept of "effective mechanical stiffness" according to which the parametric excitation creates a "fictitious" incremental stiffness which, if of suitable sign, supplies stability to the otherwise unstable system. Such a result

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has been obtained by the method of "direct separation of motion", introduced by Blekhman [11], according to which the solution is expressed by a superposition of a slow and a fast component.

Recently [12], an alternative justification has been provided in the framework of a Multiple Scale analysis. There, it has been shown that a zero-frequency forcing term appears in the perturbation equations as a combination of the excitation Ω and a non-zero natural frequency ω, around the resonant values $\omega, 2\omega, \omega/2$, as well far from all these resonances. Of course, the sign of this force decides on stability.

On the other hand, while the excitation could have a beneficial effect on the (statically) unstable mode, it could have a detrimental effect on the otherwise stable modes, via the classical mechanisms described by the Floquet theory (direct or combination resonances). Therefore, a complete analysis should investigate the possible loss of stability of all the modes.

Ref. [12] was devoted to specifically analyse, in this respect, an upright double pendulum, under over-critical gravitational forces. There, the authors were able to find lower and upper critical boundaries of stability in a wide frequency range, thus generalizing previous results obtained in [13,14], where use had been made of a perturbation method combined with the Floquet multiplier technique. The algorithm used in [12] was not a trivial application of the Multiple Scale Method. Indeed, it was shown that, according to the resonance to be investigated, integer or fractional power expansions of the state variables and of time must be used, and parameters properly ordered. Moreover, it was underlined, in dealing with the fractional series, that, differently from the usual applications of the method, the complementary solution of the perturbation equations cannot be disregarded, if inconsistencies have to be avoided.

This paper is aimed to further generalize the results of [12]. A general linear multi-degree-of-freedom Hamiltonian system, close to a divergence point, is considered, and parametrically excited by an arbitrary frequency Ω and small-amplitude δ. Several type of resonances are studied, in addition to the non-resonant excitation, namely: $\Omega = \omega_j,\ 2\omega_j,\ \omega_j/2,\ \omega_j + \omega_i,\ \omega_j - \omega_i$ where $\omega_j,\ \omega_i$ are any two (non-zero) natural frequencies. Among these, the combination resonances of sum of difference type, only occur in systems with at least three d.o.f., being $\omega_1 = 0$ the frequency associated with the buckling mode. Analytical conditions are found for stability of the parametrically excited system, and the beneficial/detrimental effect of the excitation is discussed. Numerical simulations are carried out on a triple-pendulum, for which the Multiple Scale solutions are compared with numerical solutions, based on the Floquet theory.

2. Problem position

Let us consider an undamped autonomous n-dof linear system, parametrically excited. The equations of motion, in non-dimensional form, read:

$$\mathbf{M}\ddot{\mathbf{q}} + \left(\mathbf{C}(p) + \delta\Omega^2 \cos\Omega t\mathbf{B}\right)\mathbf{q} = 0 \tag{1}$$

where $\mathbf{M} = \mathbf{M}^T$ is the mass matrix, $\mathbf{C} = \mathbf{C}^T$ is the stiffness matrix, depending on a load parameter p, $\mathbf{B}$ is the parametric excitation matrix, δ the parametric excitation amplitude and Ω the parametric excitation frequency; dots denote time-differentiation.

Let us assume that the unexcited system is stable at $p < p_0$ and unstable at $p > p_0$, where p_0 is the load critical value. Therefore, the eigenvalue problem:

$$(\mathbf{C}_0 - \omega_k^2\mathbf{M})\mathbf{u} = 0, \tag{2}$$

where $\mathbf{C}_0 = \mathbf{C}(p_0)$, admits the eigenfrequencies $(0, \omega_2, \ldots, \omega_n)$ (supposed distinct), with the associated (real) eigenvectors $(\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$. By defining the modal matrix $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$, it results, after normalization, that $\mathbf{U}^T\mathbf{C}\mathbf{U} = diag[0, \omega_2^2, \ldots, \omega_n^2]$ and $\mathbf{U}^T\mathbf{M}\mathbf{U} = \mathbf{I}$.

We put $\mathbf{C}(p) = \mathbf{C}_0 + \Delta p\mathbf{C}_1$, with the incremental load Δp, and perform the following rescaling: $\Delta p \to \varepsilon^2\Delta p, \delta \to \varepsilon\delta$, where ε is a perturbation parameter, so that Eq. (1) becomes:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}_0\mathbf{q} + \varepsilon\delta\Omega^2 \cos\Omega t\mathbf{B}\mathbf{q} + \varepsilon^2\Delta p\mathbf{C}_1\mathbf{q} = 0. \tag{3}$$

A steady solution to Eq. (3) will be pursued by the Multiple Scale Method (MSM) [15], by using, when appropriate, integer power expansions, or adapting fractional power algorithms, similarly to what done in [16,17].

As it is well-known, the MSM works as a reduction method, which contracts the dimension of the original dynamical system, by furnishing Amplitude Modulation Equations in the (time-dependent) amplitude of the resonant modes. Goal of the analysis is to obtain such reduced equations, in order to investigate on stability of the trivial equilibrium position.

3. Multiple scale analysis: integer power expansion

According to the Multiple Scale Method [15], we introduce several time-scales $t_j = \varepsilon^j t,\ j = 0, 1, \ldots$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} = \sum_{k=0}^{\infty}\varepsilon^k \mathrm{d}_k, \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\varepsilon^{k+j}\mathrm{d}_k\mathrm{d}_j \tag{4}$$

where $\mathrm{d}_k = \partial/\partial t_k$. Then, we expand the configuration variables in series of integer powers of the perturbation parameter ε: