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Multiplicity of solutions for nonlinear second order impulsive differential equations with linear derivative dependence via variational methods

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ABSTRACT

This paper uses critical point theory and variational methods to investigate the multiple solutions of boundary value problems for second order impulsive differential equations. The conditions for the existence of multiple solutions are established. An example is constructed to illustrate the proposed result.

1. Introduction

Consider the following nonlinear boundary value problems for second order impulsive differential equations:

$$\begin{cases} -(p(t)u'(t))' + r(t)u'(t) + q(t)u(t) = f(t, u(t)), & a.e.\ t \in J', \\ -\Delta(p(t_j)u'(t_j)) = I_j(u(t_j)), & j = 1, 2, \ldots, n, \\ u(0) = 0, a_1 u(1) + u'(1) = 0, \end{cases} \tag{1.1}$$

where $J = [0,1]$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$, $J' = J \backslash \{t_1, \ldots, t_n\}$, $f \in C[J \times R, R]$, $p \in C^1[0,1]$, $q \in C[0,1]$, satisfying some conditions specified later, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$ for $u'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u'(t)$, $j = 1, \ldots, n$.

The theory of impulsive differential equations has been emerging as an important area of investigation in recent years. Some classical tools such as coincidence degree theory of Mawhin, fixed point theorems in cones, and the method of lower and upper solutions have been widely used to get positive solutions of impulsive differential equations. For the theory and classical results see the monographs [1–4]. Some recent development and applications of impulsive differential equations can be seen in [5–14]. We point out that in a second order differential equation $u'' = f(t, u, u')$, one usually considers impulses in the position u and the velocity u'. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no change in position [15]. The impulses only on the velocity occurs also in impulsive mechanics [16].

Recently, taking a Dirichlet problem with impulses as a model, Nieto and O'Regan [17] have shown that the impulsive problem minimize some (energy) functional, and the critical points of that functional are indeed solutions of the impulsive

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problem. Inspired by the work [17], in this paper we use critical point theory and variational methods to investigate the multiple solutions of (1.1). Our main results extend the study made in [17–21], in the sense that we deal with a class of problems that is not considered in those papers.

The rest of the paper is organized as follows: In Section 2, we give several important lemmas. The main theorems are formulated and proved in Section 3. And in Section 4, we give an example to demonstrate the application of our results.

2. Preliminaries

Let $L(t) = \int_0^t (r(s)/p(s))ds$, $0 < m \leqslant e^{-L(t)}p(t) \leqslant M$, and $q(t) - p(t) \geqslant 0$, where $t \in J$. We transform (1.1) into the following equivalent form:

$$\begin{cases} -(e^{-L(t)}p(t)u'(t))' + e^{-L(t)}q(t)u(t) = e^{-L(t)}f(t,u(t)), & a.e.\ t \in J', \\ -\Delta(p(t_j)u'(t_j)) = I_j(u(t_j)), & j = 1,2,\ldots,n, \\ u(0) = 0, \quad a_1u(1) + u'(1) = 0. \end{cases} \tag{2.1}$$

Obviously, the solutions of (2.1) are solutions of (1.1). So it suffices to consider (2.1).

We now state some celebrated results in nonlinear functional analysis and critical point theory. Suppose that X is a Banach space (in particular a Hilbert space) and $\varphi : X \to R$ is differentiable. We say that φ satisfies the Palais–Smale condition if every sequence $\{u_k\}$ in the space X such that $\{\varphi(u_k)\}$ is bounded and $\lim_{k\to\infty}\varphi'(u_k) = 0$ contains a convergent subsequence.

Lemma 2.1 (*Mountain Pass Theorem; Theorem 4.10 in [22]*). *Let $\varphi \in C^1(X,R)$. Assume that there exist u_0, $u_1 \in X$ and a bounded neighborhood Ω of u_0 such that u_1 is not in Ω and*

$$\inf_{v\in\partial\Omega} \varphi(v) > \max\{\varphi(u_0), \varphi(u_1)\}.$$

Then there exists a critical point u of φ i.e., $\varphi'(u) = 0$, with

$$\varphi(u) > max\{\varphi(u_0), \varphi(u_1)\}.$$

Note that if either u_0 or u_1 is a critical point of φ then we obtain the existence of at least two critical points for φ.

Lemma 2.2 (*Theorem 38.A in [23]*). *For the functional $F : M \subseteq X \to R$ with M not empty, $min_{u\in M}F(u) = a$ has a solution in case the following hold:*

(i) X is a real reflexive Banach space;
(ii) M is bounded and weak sequentially closed;
(iii) F is weak sequentially lower semi-continuous on M, i.e., by definition, for each sequence $\{u_k\}$ in M such that $u_k \rightharpoonup u$ as $k \to \infty$, we have $F(u) \leqslant \underline{lim}_{k\to\infty}F(u_k)$.

Lemma 2.3 (*[Theorem 9.12 in [24]*). *Let E be an infinite dimensional real Banach space and $\varphi \in C^1(E,R)$ be even, satisfying the Palais–smale condition and $\varphi(0) = 0$. If $E = V \oplus X$, where V is finite dimensional, and φ satisfies the following conditions:*

(i) There exist constants ρ, $\sigma > 0$ such that $\varphi|_{\partial B_\rho \cap X} \geqslant \sigma$;
(ii) For each finite dimensional subspace $V_1 \subset E$, there is an $R = R(V_1)$ such that $\varphi(u) \leqslant 0$ for every $u \in V_1$ with $\|u\| > R$.

Then φ has an unbounded sequence of critical values.

Consider the Hilbert spaces $H = \{u \in H^1(0,1) : u(0) = 0\}$ with the inner product and norm

$$(u,v) = \int_0^1 e^{-L(t)}p(t)(u'(t)v'(t) + u(t)v(t))dt,$$

$$\|u\| = \left(\int_0^1 e^{-L(t)}p(t)\left(|u\prime(t)|^2 + |u(t)|^2\right)dt\right)^{\frac{1}{2}}.$$

Lemma 2.4. *Let $u \in H$ and $\|u\|_0 = max_{t\in[0,1]}|u(t)|$. Then*

$$\|u\|_0 \leqslant \frac{1}{\sqrt{m}}\|u\|.$$

Proof. The result is followed by the following inequality chain:

$$|u(t)| \leqslant \int_0^1 |u'(t)|dt \leqslant \left(\int_0^1 \frac{1}{e^{-L(t)}p(t)}dt\right)^{\frac{1}{2}}\left(\int_0^1 e^{-L(t)}p(t)|u'(t)|^2dt\right)^{\frac{1}{2}} \leqslant \frac{1}{\sqrt{m}}\|u\|.$$