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Short communication

Lie symmetry analysis of electron–electromagnetic wave interaction under condition of the anomalous Doppler effect

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ABSTRACT

Lie symmetry analysis is applied for a problem of interaction of electron cyclotron oscillators with a slow electromagnetic wave under condition of the anomalous Doppler effect. This analysis reveals scaling invariance of the system and existence of self-similar solutions which describe amplification of a short electromagnetic pulse with its subsequent compression. The results of theoretical analysis are confirmed by numerical simulations.

1. Introduction

Lie symmetry analysis is a powerful tool for finding exact solutions of nonlinear problems [1–4]. Many examples of application to physical problems have been demonstrated. Unfortunately, the benefits of this formalism are still undervalued in microwave electronics where one have to deal with various nonlinear problems of electron beam — electromagnetic wave interaction. In this article, Lie symmetry analysis of beam–wave interaction under condition of the anomalous Doppler (AD) effect is presented. Exact self-similar solutions describing amplification of a short electromagnetic pulse with subsequent compression are derived and analyzed.

The AD effect occurs when an electromagnetic oscillator moves with a superluminal velocity. In such a situation, emission of radiation is accompanied by transition of the oscillator to a higher energy level and simultaneous deceleration of its axial motion [5–7]. AD radiation of electron cyclotron oscillators is of particular interest in microwave electronics. In Fig. 1, schematic drawing of an anomalous Doppler cyclotron resonance maser (AD–CRM) is presented. The electron beam drifting along the external constant magnetic field B_0 with the constant velocity $v_\parallel$ interacts with the traveling electromagnetic wave (EMW) under the AD cyclotron resonance condition

$$\omega - kv_\parallel = -\omega_H, \tag{1}$$

where $\omega_H = eB_0/m\gamma$ is the electron cyclotron frequency, e, m, and γ are the electron charge, mass, and Lorentz-factor, respectively. From Eq. (1) it is clear that the resonance condition holds true when $v_\parallel$ is larger than the phase velocity $v_{ph} = \omega/k$ of the slow EMW. Either dielectric-loaded or periodically corrugated metallic waveguide may be used to retard the EMW.

The electron beam is initially rectilinear, i.e. the electrons are injected in the waveguide having only axial velocity. During the interaction with the EMW, the electrons lose their translational energy and gain the rotational one. In such a case, a

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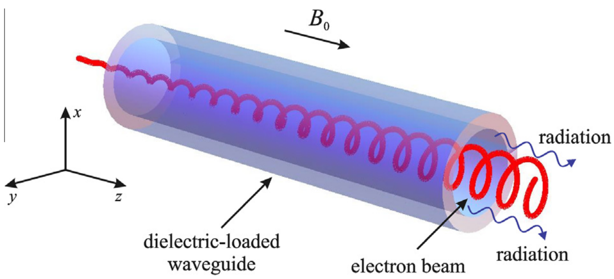

Fig. 1. Scheme of the anomalous Doppler cyclotron resonance maser.

negative-energy slow cyclotron wave [7] is excited in the electron beam. Operation of the AD–CRM was theoretically studied and experimentally demonstrated in microwave band (see e.g. Refs. [8–12]).

The electron-wave interaction in the AD–CRM is described by the following system of partial differential equations (PDEs) for normalized complex amplitude of the circularly-polarized EMW, $a(z,t) = a_x + ia_y$, and the normalized transverse momentum of the electrons, $p(z,t) = p_x + ip_y$ [8,14]:

$$\begin{aligned} &\frac{\partial a}{\partial z} + \frac{\partial a}{\partial t} = p, \\ &\frac{\partial p}{\partial z} + ip|p|^2 = a. \end{aligned} \tag{2}$$

All the variables in Eq. (2) are dimensionless, see [8,14] for details.

2. Lie symmetry analysis

For convenience, let us perform a transition to the moving reference frame by the substitution $z \to z - t$ and rewrite Eq. (2) as

$$\begin{aligned} &a_t = p, \\ &p_z + ip|p|^2 = a. \end{aligned} \tag{3}$$

Hereinafter $a_t \equiv \partial a/\partial t$, $a_z \equiv \partial a/\partial z$, etc.

Following [1–4], define the infinitesimal group generator

$$\begin{aligned} X = {} & \xi_t(z,t,a,p,a^*,p^*)\frac{\partial}{\partial t} + \xi_z(z,t,a,p,a^*,p^*)\frac{\partial}{\partial z} + \eta_a(z,t,a,p,a^*,p^*)\frac{\partial}{\partial a} + \eta_a^*(z,t,a,p,a^*,p^*)\frac{\partial}{\partial a^*} + \eta_p(z,t,a,p,a^*,p^*)\frac{\partial}{\partial p} \\ & + \eta_p^*(z,t,a,p,a^*,p^*)\frac{\partial}{\partial p^*}, \end{aligned} \tag{4}$$

which determines point symmetries of the system (3), and the extended generator

$$\begin{aligned} \tilde{X} = X & + \zeta_{at}(z,t,a,p,a^*,p^*)\frac{\partial}{\partial a_t} + \zeta_{at}^*(z,t,a,p,a^*,p^*)\frac{\partial}{\partial a_t^*} + \zeta_{az}(z,t,a,p,a^*,p^*)\frac{\partial}{\partial a_t} + \zeta_{az}^*(z,t,a,p,a^*,p^*)\frac{\partial}{\partial a_t^*} \\ & + \zeta_{pt}(z,t,a,p,a^*,p^*)\frac{\partial}{\partial p_t} + \zeta_{pt}^*(z,t,a,p,a^*,p^*)\frac{\partial}{\partial p_t^*} + \zeta_{pz}(z,t,a,p,a^*,p^*)\frac{\partial}{\partial p_z} + \zeta_{pz}^*(z,t,a,p,a^*,p^*)\frac{\partial}{\partial p_z^*}, \end{aligned} \tag{5}$$

where ζ_{at}, ζ_{az}, ζ_{pt}, ζ_{pz} are defined by the prolongation formulas [1–4]. However, $\zeta_{az} = \zeta_{pt} = 0$ because Eq. (3) do not contain the derivatives a_z, p_t. Note that $\xi_{z,t}$ are real functions, while $\eta_{a,p}$, ζ_{at}, and ζ_{pz} are complex; "*" denotes complex conjugate.

Action of the operator (5) on Eq. (3) yields the determining equations

$$\begin{aligned} &\zeta_{at} = \eta_p, \\ &\zeta_{pz} = -2ipp^*\eta_p - ip^2\eta_p^* + \eta_a. \end{aligned} \tag{6}$$

Substituting ζ_{at}, ζ_{pz} and a_t, p_z from Eq. (3) into Eq. (6) yields the over determined system of linear PDEs

$$\begin{aligned} &\frac{\partial \xi_z}{\partial t} = -p\frac{\partial \xi_z}{\partial a} - p^*\frac{\partial \xi_z}{\partial a^*}, \\ &\frac{\partial \xi_t}{\partial z} = -\left(a - ip^2p^*\right)\frac{\partial \xi_t}{\partial p} - \left(a^* + ipp^{*2}\right)\frac{\partial \xi_t}{\partial p^*}, \end{aligned} \tag{7}$$

$$\frac{\partial \xi_z}{\partial p} = \frac{\partial \xi_z}{\partial p^*} = 0, \quad \frac{\partial \xi_t}{\partial a} = \frac{\partial \xi_t}{\partial a^*} = 0, \tag{8}$$